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Two canonical passive state/signal shift realizations of passive discrete time behaviors

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Abstract

A discrete time invariant linear state/signal system Σ with a Hilbert state space $\mathcal{X}$ and a Kreĭn signal space $\mathcal{W}$ has trajectories $(x(\cdot), w(\cdot))$ that are solutions of the equation $x(n+1) = F\left(\left[\begin{smallmatrix} x(n) \\ u(n) \end{smallmatrix}\right]\right)$, where F is a bounded linear operator from $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ into $\mathcal{X}$ with a closed domain whose projection onto $\mathcal{X}$ is all of $\mathcal{X}$. This system is passive if the graph of F is a maximal nonnegative subspace of the Kreĭn space $-\mathcal{X}[+]\mathcal{X}[+]\mathcal{W}$. The future behavior $\mathfrak{W}_{\text{fut}}$ of a passive system Σ is the set of all signal components $w(\cdot)$ of trajectories $(x(\cdot), w(\cdot))$ of Σ on $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$ with $x(0) = 0$ and $w(\cdot) \in \ell^2(\mathbb{Z}^+; \mathcal{W})$. This is always a maximal nonnegative shift-invariant subspace of the Kreĭn space $k^2(\mathbb{Z}^+; \mathcal{W})$, i.e., the space $\ell^2(\mathbb{Z}^+; \mathcal{W})$ endowed with the indefinite inner product inherited from $\mathcal{W}$. Subspaces of $k^2(\mathbb{Z}^+; \mathcal{W})$ with this property are called passive future behaviors. In this work we study passive state/signal systems and passive behaviors (future, full, and past). In particular, we define and study the input and output maps of a passive state/signal system, and the past/future map of a passive behavior. We then turn to the inverse problem, and construct two passive state/signal realizations of a given passive future behavior $\mathfrak{W}_+$, one of which is observable and backward conservative, and the other controllable and forward conservative. Both of these are canonical in the sense that they are uniquely determined by the given data $\mathfrak{W}_+$, in contrast earlier realizations that depend not only on $\mathfrak{W}_+$, but also on some arbitrarily chosen fundamental decomposition of the signal space $\mathcal{W}$. From

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our canonical realizations we are able to recover the two standard de Branges–Rovnyak input/state/output shift realizations of a given operator-valued Schur function in the unit disk.

Keywords: Passive; Conservative; Behavior; State/signal; De Branges–Rovnyak model; Input map; Output map; Past/future map; Kreĭn space; Quotient space

Contents

1. Introduction

In this work we continue our study of passive linear discrete time invariant s/s (state/signal) system begun in [2–5]. However, the approach taken here is somewhat different from the approach in [2–5], and the present article is essentially self-contained.

The s/s systems theory differs from the standard i/s/o (input/state/output) systems theory in the sense that no distinction is made between input and output signals, only between an "internal" state $x \in \mathcal{X}$ and an "external" interaction signal $w \in \mathcal{W}$. In [2] it was assumed that both the state space $\mathcal{X}$ and the signal space $\mathcal{W}$ are Hilbert spaces, but in the subsequent articles [3–5] dealing with passive systems the signal space $\mathcal{W}$ was replaced by a Kreĭn space (the state space $\mathcal{X}$ still remains a Hilbert space).

A trajectory $(x(\cdot), w(\cdot))$ of a linear discrete time-invariant s/s system Σ on a discrete time interval $I \subset \mathbb{Z}$ consists of an $\mathcal{X}$-valued state sequence $x(\cdot)$ and a $\mathcal{W}$-valued signal sequence $w(\cdot)$ satisfying the equations

$$x(n+1) = F \begin{bmatrix} x(n) \\ w(n) \end{bmatrix}, \quad n \in I, \tag{1.1}$$

where F is a bounded linear operator with closed domain $\mathcal{D}(F) \subset \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ and values in $\mathcal{X}$ with the property that the projection of $\mathcal{D}(F)$ onto $\mathcal{X}$ is all of $\mathcal{X}$. The last property is equivalent to the following property of the set of trajectories of Σ: for every discrete time interval I with finite left end-point m and for every $x_m \in \mathcal{X}$ there exists at least one trajectory $(x(\cdot), w(\cdot))$ of Σ on I with initial state $x(m) = x_m$. Earlier in [2–5] we primarily restricted our attention to the interval $I = \mathbb{Z}^+ := \{k \in \mathbb{Z} \mid k \geqslant 0\}$, but below we shall, in addition, consider the cases $I = \mathbb{Z}$ and $I = \mathbb{Z}^- := \{k \in \mathbb{Z} \mid k < 0\}$, and occasionally some other intervals.

A s/s system is called forward passive if, for every discrete time interval I and every trajectory $(x(\cdot), w(\cdot))$ of Σ in I, it is true that

$$-\|x(n+1)\|_{\mathcal{X}}^2 + \|x(n)\|_{\mathcal{X}}^2 + [w(n), w(n)]_{\mathcal{W}} \geqslant 0, \quad n \in I, \tag{1.2}$$

where $\|\cdot\|_{\mathcal{X}}$ is the norm in the Hilbert space $\mathcal{X}$ and $[\cdot,\cdot]_{\mathcal{W}}$ is the inner product in the Kreĭn space $\mathcal{W}$. In view of the time-invariance of (1.1), it is enough that property (1.2) holds on the interval $I = \{0\}$. This property can be dressed in a geometric form in terms of the Kreĭn (node) space $\mathfrak{K} := -\mathcal{X} [\dotplus] \mathcal{X} [\dotplus] \mathcal{W}$ as follows: condition (1.2) holds if and only if the graph V of the operator F in (1.1) is a nonnegative subspace of $\mathfrak{K}$. By replacing F in (1.1) by its graph V we can rewrite (1.1) in the equivalent form

$$\begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \quad n \in I. \tag{1.3}$$

The subspace V above is called the generating subspace of Σ, since condition (1.3) defines the set of all trajectories $(x(\cdot), w(\cdot))$ of Σ on any interval I.

The above discussion can be summarized as follows. By a linear discrete time-invariant s/s system we mean a colligation $\Sigma = (V; \mathcal{X}, \mathcal{W})$, where $\mathcal{X}$ is a Hilbert (state) space, $\mathcal{W}$ is a Kreĭn (signal) space, and V is a generating subspace of the Kreĭn (node) space $\mathfrak{K} = -\mathcal{X} [\dotplus] \mathcal{X} [\dotplus] \mathcal{W}$, i.e., a subspace which is the graph of an operator F with the properties described in the connection with (1.1).

Given a s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$, there is another s/s system $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$, called the adjoint of Σ, where $\mathcal{W}_* = -\mathcal{W}$ (this is the same space as $\mathcal{W}$ but with the inner product $[\cdot,\cdot]_{-\mathcal{W}} = -[\cdot,\cdot]_{\mathcal{W}}$), and

$$V_* = \begin{bmatrix} 0 & 1_{\mathcal{X}} & 0 \\ 1_{\mathcal{X}} & 0 & 0 \\ 0 & 0 & 1_{[\mathcal{W}_*,\mathcal{W}]} \end{bmatrix} V^{[\perp]}, \tag{1.4}$$

where $V^{[\perp]}$ is the orthogonal companion to V in $\mathfrak{K}$, and $1_{[\mathcal{W}_*,\mathcal{W}]}$ is the identity map from $\mathcal{W}$ to $\mathcal{W}_*$. The system Σ is called backward passive if Σ_* is forward passive, and Σ is called *passive* if it is both forward and backward passive. This implies that if a s/s system Σ is passive, then its generating subspace V is a *maximal nonnegative subspace of the node space* $\mathfrak{K}$.

Conversely, suppose that V is an arbitrary maximal nonnegative subspace of $\mathfrak{K}$. Let $\mathcal{W} = -\mathcal{Y} [\dotplus] \mathcal{U}$ be a fundamental decomposition of $\mathcal{W}$ (i.e., $\mathcal{Y}$ and $\mathcal{U}$ are Hilbert spaces, and the sum is orthogonal). Then, by standard Kreĭn space theory, V has a graph representation of the type

$$V = \left\{ \begin{bmatrix} Ax + Bu \\ x \\ Cx + Du \\ u \end{bmatrix} \;\middle|\; x \in \mathcal{X} \text{ and } u \in \mathcal{U} \right\}, \tag{1.5}$$

where $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a linear contraction $\mathcal{X} \oplus \mathcal{U} \to \mathcal{X} \oplus \mathcal{Y}$. This means that V is the graph of the operator F defined by

$$F\begin{bmatrix} x_0 \\ y_0 \\ u_0 \end{bmatrix} = Ax_0 + Bu_0, \qquad \mathcal{D}(F) = \left\{ \begin{bmatrix} x_0 \\ y_0 \\ u_0 \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \\ \mathcal{U} \end{bmatrix} \;\middle|\; y_0 = Cx_0 + Du_0 \right\}.$$

Trivially, this operator F satisfies the conditions listed below (1.1), and hence $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is a s/s system. This system is passive since V is maximal nonnegative. Thus, we conclude that V is the generating subspace of a passive s/s system if and only if V is maximal nonnegative in the node space $\mathfrak{K}$. In this article we discuss only passives s/s systems.

In the terminology of [2,3], the existence of the graph representation (1.5) means that every fundamental decomposition of $\mathcal{W}$ is admissible for the passive s/s system Σ. The corresponding i/s/o system $\Sigma_{\text{i/s/o}} = ([\begin{smallmatrix} A & B \\ C & B \end{smallmatrix}]; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is called a (scattering) *i/s/o representation of* Σ. If we decompose the signal $w(\cdot)$ in (1.3) into $w(\cdot) = u(\cdot) + y(\cdot)$, where the values of $u(\cdot)$ and $y(\cdot)$ lie in $\mathcal{U}$ and $\mathcal{Y}$, respectively, then (1.3) takes the form

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n), \\ y(n) &= Cx(n) + Du(n), \quad n \in I. \end{aligned} \tag{1.6}$$

See [2,3] for more details.

Since every Kreĭn space $\mathcal{W}$ that is neither a Hilbert space nor an anti-Hilbert spaces has infinitely many fundamental decompositions, this means that a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ with a Kreĭn signal space $\mathcal{W}$ usually has an infinite family $\Sigma_{\text{i/s/o}} = ([\begin{smallmatrix} A & B \\ C & B \end{smallmatrix}]; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ of scattering i/s/o representations (in the exceptional cases $\Sigma_{\text{i/s/o}}$ is unique, but it has no input or no output). Each such system $\Sigma_{\text{i/s/o}}$ has a scattering matrix $\widehat{\mathfrak{D}}(z) = zC(1 - zA)^{-1}B + D$ which is a Schur class function, i.e., a $\mathcal{B}(\mathcal{U}; \mathcal{Y})$-valued analytic contractive function in the unit disk. This function has a power series expansion $\widehat{\mathfrak{D}}(z) = \sum_{k=0}^{\infty} D(k)z^k$ with contractive coefficients $D(k) \in \mathcal{B}(\mathcal{U}; \mathcal{Y})$. Different choices of the fundamental decomposition gives different systems $\Sigma_{\text{i/s/o}}$ and different scattering matrices. Using the coefficients $D(k)$ of each scattering matrix $\widehat{\mathfrak{D}}(z)$ we can define a block-Toeplitz operator $\mathfrak{D} : \ell^2(\mathcal{U}) \to \ell^2(\mathcal{Y})$ by

$$(\mathfrak{D}u)(n) = \sum_{k=-\infty}^{n} D(n-k)u(k), \quad n \in \mathbb{Z},\ u(\cdot) \in \ell^2(\mathbb{Z}; \mathcal{U}),$$

and we can also define two additional block Toeplitz operators $\mathfrak{D}_+ : \ell^2(\mathbb{Z}^+; \mathcal{U}) \to \ell^2(\mathbb{Z}^+; \mathcal{Y})$ and $\mathfrak{D}_- : \ell^2(\mathbb{Z}^-; \mathcal{U}) \to \ell^2(\mathbb{Z}^-; \mathcal{Y})$ by $\mathfrak{D}_+ := \mathfrak{D}|_{\ell^2(\mathbb{Z}^+; \mathcal{U})}$ and $\mathfrak{D}_- := P_{\ell^2(\mathbb{Z}^-; \mathcal{Y})}\mathfrak{D}|_{\ell^2(\mathbb{Z}^-; \mathcal{U})}$. A crucial fact is that although $\mathfrak{D}$, $\mathfrak{D}_+$, and $\mathfrak{D}_-$ do depend on the fundamental decomposition $\mathfrak{W} = -\mathcal{Y} [\dot{+}] \mathcal{U}$, the graphs of these three operators do not. We call these three graphs the *full, future, and past behaviors*, respectively, of Σ.

Above we defined the full, future and past behaviors of a passive s/s system Σ in terms of an i/s/o representation of Σ, but they can also be defined directly by means of trajectories of Σ. To do this we first need to introduce the notion of an externally generated stable trajectory of a passive s/s system. A trajectory $(x(\cdot), w(\cdot))$ of Σ on a discrete time interval I is called *stable* if

$$x(\cdot) \in \ell^\infty(I; \mathcal{X}) \quad \text{and} \quad w(\cdot) \in \ell^2(I; \mathcal{W}) \tag{1.7}$$

(see Section 2 for details). If $(x(\cdot), w(\cdot))$ is a trajectory of Σ on I, then by (1.2),

$$\|x(n+1)\|_{\mathcal{X}}^2 \leqslant \|x(m)\|_{\mathcal{X}}^2 + \sum_{k=m}^{n} [w(k), w(k)]_{\mathcal{W}}, \quad m, n \in I,\ m \leqslant n. \tag{1.8}$$

Thus, if I is an interval with finite left end-point m, then the first condition $x(\cdot) \in \ell^\infty(I; \mathcal{X})$ in (1.7) is implied by the second condition $w(\cdot) \in \ell^2(I; \mathcal{W})$, so to guarantee the stability of the trajectory it suffices to require that $w(\cdot) \in \ell^2(I; \mathcal{W})$. If $x(m) = 0$, then we call this trajectory *externally generated*. If the left end-point of the interval I is $-\infty$, then we call a trajectory externally generated if $x(m) \to 0$ in $\mathcal{X}$ as $m \to -\infty$. Also such a trajectory is stable if and only if $w(\cdot) \in \ell^2(I; \mathcal{W})$; this follows from (1.8) by letting $m \to -\infty$.

The sum in (1.8) (where we allow $m = -\infty$ or $n = \infty$ or both) can be interpreted as an indefinite inner product in $\ell^2([m,n]; \mathcal{W})$, where $[m,n] := \{k \in \mathbb{Z} \mid m \leqslant k \leqslant n\}$ (and we replace "$\leqslant$" by "$<$" if $m = -\infty$ or $n = \infty$). By $k^2(I; \mathcal{W})$ we denote the space $\ell^2(I; \mathcal{W})$ equipped with the indefinite inner product

$$[w_1(\cdot), w_2(\cdot)]_{k^2(I;\mathcal{W})} = \sum_{k \in I} [w(k), w(k)]_{\mathcal{W}}. \tag{1.9}$$

It is easy to see that this is a Kreĭn space. We shall make frequent use of the special time intervals $\mathbb{Z}^+$, $\mathbb{Z}$, and $\mathbb{Z}^-$, and therefore abbreviate $k^2_+(\mathcal{W}) := k^2(\mathbb{Z}^+; \mathcal{W})$, $k^2(\mathcal{W}) := k^2(\mathbb{Z}; \mathcal{W})$, and $k^2_-(\mathcal{W}) := k^2(\mathbb{Z}^-; \mathcal{W})$.

By the future, full, and past behaviors of the passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ we mean the set of all the signal parts $w(\cdot)$ of all the externally generated stable trajectories $(x(\cdot), w(\cdot))$ on $\mathbb{Z}^+$, $\mathbb{Z}$, and $\mathbb{Z}^-$, respectively. We often denote these three sets by $\mathfrak{W}^\Sigma_{\text{fut}}$, $\mathfrak{W}^\Sigma_{\text{full}}$, and $\mathfrak{W}^\Sigma_{\text{past}}$, respectively. (Earlier, in [3], we have studied possibly non-stable future behaviors of Σ and called these simply "behaviors".) It turns out that the maximal nonnegativity of V in $\mathfrak{K}$ implies that $\mathfrak{W}^\Sigma_{\text{fut}}$, $\mathfrak{W}^\Sigma_{\text{full}}$, and $\mathfrak{W}^\Sigma_{\text{past}}$ are maximal nonnegative subsets of $k^2_+(\mathcal{W})$, $k^2(\mathcal{W})$, and $k^2_-(\mathcal{W})$, respectively, with some additional properties that we shall describe next.

Because of the time-invariance of (1.3), if we shift a trajectory of Σ left or right, then it is still a trajectory of Σ (on a new shifted interval). This implies that $\mathfrak{W}^\Sigma_{\text{fut}}$, $\mathfrak{W}^\Sigma_{\text{full}}$, and $\mathfrak{W}^\Sigma_{\text{past}}$ are shift-invariant in the following sense. Let us denote the standard right-shift operators in $k^2_+(\mathcal{W})$, $k^2(\mathcal{W})$, and $k^2_-(\mathcal{W})$ by S_+, S, and S_-, respectively. Then that $\mathfrak{W}^\Sigma_{\text{fut}}$ is S_+-invariant, $\mathfrak{W}^\Sigma_{\text{full}}$ is S-reducing (it is invariant under both S and S^{-1}), and $\mathfrak{W}^\Sigma_{\text{past}}$ is S_--invariant. In addition, $\mathfrak{W}^\Sigma_{\text{full}}$ has one extra property, called *causality* (see Section 2 for the exact definition). It turns out that there is a one-to-one correspondence between the three sets $\mathfrak{W}^\Sigma_{\text{fut}}$, $\mathfrak{W}^\Sigma_{\text{full}}$, and $\mathfrak{W}^\Sigma_{\text{past}}$: it is possible to construct natural maps that take $\mathfrak{W}^\Sigma_{\text{fut}}$ one-to-one onto $\mathfrak{W}^\Sigma_{\text{full}}$ and $\mathfrak{W}^\Sigma_{\text{full}}$ one-to-one onto $\mathfrak{W}^\Sigma_{\text{past}}$.

Since the future, full, and past behaviors induced by a passive s/s system have the properties described above, we use this fact as a motivation to introduce the following notions: by a *passive future behavior* $\mathfrak{W}_{\text{fut}}$ on the Kreĭn signal space $\mathcal{W}$ we mean a maximal nonnegative S_+-invariant subspace of $k^2_+(\mathcal{W})$, by a *passive full behavior* $\mathfrak{W}_{\text{full}}$ on $\mathcal{W}$ we mean a maximal nonnegative S-reducing causal subspace of $k^2(\mathcal{W})$, and by a *passive past behavior* $\mathfrak{W}_{\text{past}}$ on $\mathcal{W}$ we mean a maximal nonnegative S_--invariant subspace of $k^2_-(\mathcal{W})$.

The theory which we have summarized above is developed in full detail in Section 2. Adjoint systems and behaviors, as well as anti-passive reflected s/s systems are studied in Section 3. In Section 4 we present two Hilbert spaces $\mathcal{H}(\mathfrak{W}_+)$ and $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ that play fundamental roles in the remainder of this article. Here $\mathcal{H}(\mathfrak{W}_+)$ is the subspace of the quotient $k^2_+(\mathcal{W})/\mathfrak{W}_+$ consisting of all those equivalence classes whose $\mathcal{H}(\mathfrak{W}_+)$-norm, defined in (4.17) below, is finite. The Hilbert space $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ is constructed in a similar way, with $\mathfrak{W}_+$ replaced by the orthogonal companion to a passive past behavior $\mathfrak{W}_-$, interpreted as a maximal nonnegative subspace of $-k_-^2(\mathcal{W})$. Both of these spaces are special cases of the spaces $\mathcal{H}(\mathcal{Z})$ introduced and studied in [6], where $\mathcal{Z}$ is a maximal nonnegative subspace of a Kreĭn space $\mathcal{X}$. A short review of the spaces $\mathcal{H}(\mathcal{Z})$ is given in Section 4, including the descriptions and properties of the two spaces $\mathcal{H}(\mathfrak{W}_+)$ and $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$.

In Section 5 we develop the passive s/s systems theory further and introduce the input map $\mathfrak{B}_\Sigma$ and the output map $\mathfrak{C}_\Sigma$ of a passive s/s system Σ. Here $\mathfrak{B}_\Sigma$ is a contraction from $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ to $\mathcal{X}$, which is the unique extension to $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ of the map from the signal part $w(\cdot)$ of an externally generated trajectory $(x(\cdot), w(\cdot))$ on $\mathbb{Z}^-$ to $x(0)$, whereas $\mathfrak{C}_\Sigma$ is a contraction from $\mathcal{X}$ to $\mathcal{H}(\mathfrak{W}_+)$, which is equal to the map from the initial state $x(0)$ of a stable trajectory $(x(\cdot), w(\cdot))$ on $\mathbb{Z}^+$ to its signal part $w(\cdot)$ factored over the future behavior $\mathfrak{W}_+$. In Section 6 we introduce the past/future map $\Gamma_{\mathfrak{W}}$ of a passive full behavior $\mathfrak{W}$. This map plays a decisive role in our study of the inverse problem described below. It is a contraction from $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ to $\mathcal{H}(\mathfrak{W}_+)$, and it is the unique extension of the map from the past behavior $\mathfrak{W}_-$ to the restriction of the full behavior $\mathfrak{W}$ to $\mathbb{Z}^+$ factored over the future behavior $\mathfrak{W}_+$. Moreover, $\Gamma_{\mathfrak{W}} = \mathfrak{C}_\Sigma \mathfrak{B}_\Sigma$ whenever Σ is a passive s/s system with full behavior $\mathfrak{W}$.

Sections 7 and 8 are devoted to the so called *inverse problem*: given a passive future, full, or past behavior, find a passive s/s system Σ with some appropriate extra properties (that will be discussed in the next two paragraphs) whose future, full, or past behavior coincides with the given behavior. This is the s/s analogue of the inverse problem in i/s/o system theory (in scattering form): find a (scattering) passive i/s/o system whose transfer function (scattering matrix) is equal to a given Schur class function.

In order to give a more complete description of the inverse problem we need to introduce some more notions. A s/s system Σ is *forward conservative* if (1.2) holds in the form of an equality for all trajectories of Σ, and it is *backward conservative* if the adjoint system Σ_* is forward conservative. Thus, $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is passive and forward conservative if and only if V is maximal nonnegative and $V \subset V^{[\perp]}$ (this inclusion means that V is neutral), and Σ is passive and backward conservative if and only if V is maximal nonnegative and $V^{[\perp]} \subset V$. Both of these conditions hold if and only if V is a Lagrangian subspace of $\mathfrak{K}$, in which case Σ is called *conservative*. For a conservative system the inequality (1.2) holds in the form of an equality, both for the original system and for the adjoint s/s system.

The subspace of $\mathcal{X}$ that we get by taking the closure in $\mathcal{X}$ of all states $x(n)$ that appear in externally generated trajectories $(x(\cdot), w(\cdot))$ of Σ on $\mathbb{Z}^+$ is called the (approximately) *reachable subspace*, and we denote it by $\mathfrak{R}_\Sigma$. If $\mathfrak{R}_\Sigma = \mathcal{X}$, then Σ is called *controllable*. The subspace of all $x_0 \in \mathcal{X}$ with the property that there exists some trajectory $(x(\cdot), w(\cdot))$ of Σ on $\mathbb{Z}^+$ with $x(0) = x_0$ for which w vanishes identically is called the *unobservable subspace*, and it is denoted by $\mathfrak{U}_\Sigma$. If $\mathfrak{U}_\Sigma = \{0\}$, then Σ is called (approximately) *observable*. A s/s system Σ is called *simple* if $\mathcal{X} = \overline{\mathfrak{R}_\Sigma + \mathfrak{U}_\Sigma^\perp}$, or equivalently, if $\mathfrak{U}_\Sigma \cap \mathfrak{R}_\Sigma^\perp = \{0\}$, and it is *minimal* if it is both controllable and observable.

The following solution to the inverse problem can be derived from the proof of [3, Theorem 8.6].

Theorem 1.1. *Let $\mathcal{W}$ be a Kreĭn space, and let $\mathfrak{W}_+$ be an arbitrary maximal nonnegative S_+-invariant subspace of the Kreĭn space $k_+^2(\mathcal{W})$. Then there exist four passive s/s systems Σ_{obc}, Σ_{cfc}, Σ_{sc}, and $\Sigma_{\min}$ with future behavior $\mathfrak{W}_+$ satisfying the following additional conditions*:

(1) Σ_{obc} *is observable and backward conservative*;
(2) Σ_{cfc} *is controllable and forward conservative*;
(3) Σ_{sc} *is simple and conservative*;
(4) $\Sigma_{\min}$ *is minimal.*

The s/s systems Σ_{obc}, Σ_{cfc}, and Σ_{sc} are uniquely defined by $\mathfrak{W}_+$ up to unitary similarity, and Σ_{sc} and $\Sigma_{\min}$ can be obtained by dilations and compressions, respectively, from Σ_{obc} and Σ_{cfc}.

The notion of unitary similarity of s/s systems used above is defined in a natural way; see Definition 7.6 below.

In Sections 7 and 8 we present special realizations of types (1) and (2) of a given future behavior $\mathfrak{W}_+$. These realizations are canonical in the sense that they are uniquely determined by the given data $\mathfrak{W}_+$, in contrast to the realizations given in [3] that depend not only on $\mathfrak{W}_+$, but also on some arbitrarily chosen fundamental decomposition of the signal space $\mathcal{W}$. The state space in the first canonical model is $\mathcal{H}(\mathfrak{W}_+)$, and the state space in the second canonical model is $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$. We shall return elsewhere to the questions of how to construct special canonical realizations of the types (3) and (4).

Finally, in Sections 9 and 10 we explain the relationship between our two canonical models and the two canonical i/s/o de Branges–Rovnyak scattering models whose scattering matrices coincide with a given Schur function Φ in the unit disk. This involves mapping the space $\mathcal{H}(\mathcal{Z})$ (where $\mathcal{Z}$ is either $\mathfrak{W}_+$ of $\mathfrak{W}_-^{[\perp]}$) onto a de Branges complementary space $\mathcal{H}(A)$. The general construction is of the following type (see Section 9 for more details). Let $\mathcal{Z}$ be a maximal nonnegative subspace of a Kreĭn space $\mathcal{X}$, and fix some fundamental decomposition $\mathcal{X} = -\mathcal{Y}[\dotplus]\mathcal{U}$. Then, with respect to this decomposition, $\mathcal{Z}$ is the graph of a linear contraction $A : \mathcal{U} \to \mathcal{Y}$. In [6] we showed that the mapping T from an equivalence class $h \in \mathcal{H}(A)$ containing a vector $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ onto $Th = y - Au$ is a unitary operator from $\mathcal{H}(\mathcal{Z})$ onto the de Branges complementary Hilbert space $\mathcal{H}(A)$. That space, with a suitable choice of A, was used as the state space in the two de Branges–Rovnyak models constructed in [7,8]. In operator theory these systems are called "operators nodes with a given characteristic function Φ" that are either "co-isometric and closely outer connected" or "isometric and closely inner connected", respectively. To obtain these two i/s/o models from our canonical s/s models we fix some fundamental decomposition $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ of the signal space $\mathcal{W}$, which induces the fundamental decompositions $k_\pm^2(\mathcal{W}) = -\ell_\pm^2(\mathcal{Y})\,[\dotplus]\,\ell_\pm^2(\mathcal{U})$. The operator A is replaced by either $\widehat{\mathfrak{D}}_+$ or $\widehat{\mathfrak{D}}_-^*$, where $\widehat{\mathfrak{D}}_\pm$ are the frequency domain versions of the block Toeplitz operators $\mathfrak{D}_\pm$ mentioned earlier. There is a small technical difference between the second canonical model that we obtain and the one in, e.g., [1], namely the state space of our version of this model in a subspace of the Hardy space H_-^2 defined on the *outside* of the unit disk $\mathfrak{D}_+$, whereas the state space of the standard model is a subspace of H_+^2 in the unit disk itself. However, this difference is not significant, since H_+^2 can be mapped onto H_-^2 by the unitary transformation $\hat{u}_+(z) \mapsto \hat{u}_-(z) := z^{-1}\hat{u}_+(1/z)$. (The same observation is made in [9,10], too.)

Our final formulas for the coefficients A, B, C, and D of the controllable forward conservative i/s/o model depend in a crucial way on the frequency domain input/output version $\Gamma_{(\widehat{\mathfrak{D}}_-^*,\widehat{\mathfrak{D}}_+)}$ of the past/future map $\Gamma_{\mathfrak{W}}$ mentioned earlier. The map $\Gamma_{(\widehat{\mathfrak{D}}^*_-,\widehat{\mathfrak{D}}_+)}$ is a unitary image of the operator $f(z) \mapsto \tilde{f}(z)$ in [7, Theorem 5, p. 350] and also of the operator Λ^* in [1, Theorem 3.4.1, p. 107] (the setting in [1] is slightly more general in the sense that it permits the state space to be a Pontryagin space and the scattering matrix to be a generalized Schur function).

In [9,10] Nikolskiĭ and Vasyunin present a "coordinate free" model of a simple conservative i/s/o scattering system whose scattering matrix coincides with a given Schur function. The philosophy behind the work of Nikolskiĭ and Vasyunin is very different from the philosophy underlying our work. The coordinate free Nikolskiĭ–Vasyunin model contains a "free" parameter Π, and by the appropriate choice of this parameter it is possible to recover all simple conservative shift models whose characteristic function is equal to a given Schur function φ, including the Sz.-Nagy–Foiaş model, the de Branges–Rovnyak model, and the Pavlov model. In this sense the Nikolskiĭ–Vasyunin model is "universal". On the other hand, our canonical s/s shift models are completely determined by a given future behavior, and in particular, they are "coordinate free" in the sense that they do not depend on some arbitrarily chosen fundamental decomposition $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ of the given signal space $\mathcal{W}$. Different choices of such a decomposition give rise to different graph representations of the frequency domain version of the given future behavior as the graph of multiplication operators induced by different Schur functions φ (with varying input and output spaces), and the corresponding i/s/o representations of our canonical s/s models are equivalent to the i/s/o de Branges–Rovnyak realizations of φ. Another difference between our present work and the cited work by Nikolskiĭ and Vasyunin is that their model is a simple and conservative (i/s/o) model, in contranst to our two passive s/s models, one of which is observable and backward conservative, and the other controllable and forward conservative. A canonical simple conservative s/s model also exists, and we shall return to this model elsewhere.

Notations. The following standard notations are used below. $\mathbb{C}$ is the complex plane, $\mathbb{D}_+ := \{z \in \mathbb{C} \mid |z| < 1\}$, $\mathbb{D}_- := \{z \in \mathbb{C} \mid |z| > 1\} \cup \{\infty\}$, $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$, and $\mathbb{Z}^- = \{-1, -2, -3, \ldots\}$. For any set Ω, we denote the closure of Ω by $\overline{\Omega}$, and we denote the closed linear span of a collection $\{\Omega_\alpha\}_{\alpha \in A}$ of sets in a Hilbert or Kreĭn space by $\bigvee_{\alpha \in A} \Omega_\alpha$.

The space of bounded linear operators from one Kreĭn space $\mathcal{U}$ to another Kreĭn space $\mathcal{Y}$ is denoted by $\mathcal{B}(\mathcal{U};\mathcal{Y})$. The domain, range, and kernel of a linear operator A are denoted by $\mathcal{D}(A)$, $\mathcal{R}(A)$, and $\mathcal{N}(A)$, respectively. The restriction of A to some subspace $\mathcal{Z} \subset \mathcal{D}(A)$ is denoted by $A|_{\mathcal{Z}}$. The identity operator on $\mathcal{U}$ is denoted by $1_{\mathcal{U}}$, or by 1 if the space is clear from the context. The orthogonal projection onto a closed subspace $\mathcal{Y}$ of a Kreĭn space $\mathcal{K}$ is denoted by $P_{\mathcal{Y}}$.

The inner product in a Hilbert space $\mathcal{X}$ is denoted by $(\cdot,\cdot)_{\mathcal{X}}$, and the inner product in a Kreĭn space $\mathcal{K}$ is denoted by $[\cdot,\cdot]_{\mathcal{K}}$. The orthogonal sum of $\mathcal{U}$ and $\mathcal{Y}$ is denoted by $\mathcal{U} \oplus \mathcal{Y}$ in the case of Hilbert spaces, and by $\mathcal{U}\,[\dotplus]\,\mathcal{Y}$ in the case of Kreĭn spaces. The anti-space $-\mathcal{K}$ of a Kreĭn space is algebraically the same space as $\mathcal{K}$, but it has a different inner product $[\cdot,\cdot]_{-\mathcal{K}} := -[\cdot,\cdot]_{\mathcal{K}}$.

We denote the product of two Kreĭn or Hilbert spaces $\mathcal{Y}$ and $\mathcal{U}$ by $\left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$. If $\mathcal{L}$ is a set of vectors in a Kreĭn space, then $\mathcal{L}^{[\perp]}$ is the orthogonal companion to $\mathcal{L}$, i.e.,

$$\mathcal{L}^{[\perp]} := \bigl\{x \in \mathcal{K} \bigm| [x,y]_{\mathcal{K}} = 0 \text{ for all } y \in \mathcal{L}\bigr\}.$$

If $w(\cdot)$ is a sequence with values in a Kreĭn or Hilbert space $\mathcal{W}$ defined on some discrete time interval I, then $S^{\pm 1}w$ is the sequence $w(\cdot)$ shifted one step to the right or left, respectively (this includes a right or left shift of I if $I \neq \mathbb{Z}$). For sequences $w(\cdot)$ defined on $\mathbb{Z}^+$ we define $(S_+w)(0) = 0$, $(S_+w)(n) = w(n-1)$, $n \geqslant 1$, and for sequences $w(\cdot)$ defined on $\mathbb{Z}^-$ we define $(S_-w)(n) = w(n-1)$, $n \in \mathbb{Z}^-$. If we want to emphasize that the values of w lie in $\mathcal{W}$ we write $S^{\mathcal{W}}$ instead of S.

2. Passive future, full, and past behaviors

Passive state/signal systems

A passive linear discrete time invariant s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ has a Hilbert (state) space $\mathcal{X}$, a Kreĭn (signal) space $\mathcal{W}$, and a (generating) maximal nonnegative subspace V of the Kreĭn space $\mathfrak{K} = -\mathcal{X} [\dot{+}] \mathcal{X} [\dot{+}] \mathcal{W}$. A *trajectory of Σ on a discrete time interval I* is a pair of sequences $(x(\cdot), w(\cdot))$ satisfying (1.3). Observe that $w(\cdot)$ is always defined on I, but that $x(\cdot)$ is defined at one extra point at the right end if I is bounded to the right, i.e., if $w(\cdot)$ is defined on $I = (m, n) := \{k \in \mathbb{Z} \mid m < k < n\}$, then $x(\cdot)$ is defined on $(m, n] := \{k \in \mathbb{Z} \mid m < k \leqslant n\}$ (here we allow $m = -\infty$; if $n = +\infty$, then these two sets coincide. Earlier, in [2–5], we most of the time took the interval I to be $I = \mathbb{Z}^+ = \{0, 1, 2, \dots\}$, but below we shall also consider other intervals, finite or infinite. In particular, in addition to $\mathbb{Z}^+$ we shall frequently take $I = \mathbb{Z}$ or $I = \mathbb{Z}^- = \{-1, -2, -3, \dots\}$ (in which case $x(k)$ is also defined for $k = 0$). By a *past* trajectory we mean a trajectory on $\mathbb{Z}^-$, by a *full* trajectory we mean a trajectory on $\mathbb{Z}$, and by a *future* trajectory we mean a trajectory on $\mathbb{Z}^+$. In the case where the interval I is bounded to the left we call a trajectory $(x(\cdot), w(\cdot))$ on I *externally generated* if x vanishes at the left end-point of I, i.e., $x(m) = 0$ if $I = [m, n) := \{z \in \mathbb{Z} \mid m \leqslant z < n\}$ (where we allow $n = \infty$), and if I is unbounded to the left we call the trajectory externally generated if $x(m) \to 0$ in $\mathcal{X}$ as $m \to -\infty$.

Stable trajectories of passive state/signal systems

All the s/s systems in this article will be passive. A trajectory $(x(\cdot), w(\cdot))$ of the passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ on an interval I is called *stable* if

$$w(\cdot) \in k^2(I; \mathcal{W}) \quad \text{and} \quad x(\cdot) \in \ell^\infty(I; \mathcal{X}) \tag{2.1}$$

(strictly speaking, the restriction of $x(\cdot)$ to the interval I should belong to $\ell^\infty(I; \mathcal{X})$). Here $\ell^\infty(I; \mathcal{X})$ is the Banach space of bounded $\mathcal{X}$-valued sequences on the interval I. The space $k^2(I; \mathcal{W})$ is a Kreĭn space whose inner product is defined in (2.3) below. A sequence $w(\cdot)$ with values in $\mathcal{W}$ belongs to $k^2(I; \mathcal{W})$ if and only if

$$\sum_{k \in I} \|w(k)\|_{\mathcal{W}}^2 < \infty, \tag{2.2}$$

where $\|\cdot\|_{\mathcal{W}}$ is some *admissible* Hilbert space norm in the Kreĭn space $\mathcal{W}$, given by

$$\|w\|_{\mathcal{W}}^2 = -[P_{\mathcal{W}_-} w, P_{\mathcal{W}_-} w]_{\mathcal{W}} + [P_{\mathcal{W}_+} w, P_{\mathcal{W}_+} w]_{\mathcal{W}}$$

for some fundamental decomposition $\mathcal{W} = -\mathcal{W}_- [\dot{+}] \mathcal{W}_+$ where $\mathcal{W}_-$ and $\mathcal{W}_+$ are Hilbert spaces with the norms inherited from $-\mathcal{X}$ and $\mathcal{X}$, respectively. Different fundamental decompositions give different norms $\|\cdot\|_{\mathcal{W}}$, but they are all equivalent, so (2.2) is independent of the chosen admissible norm in the sense that if (2.1) holds for one admissible norm $\|\cdot\|_{\mathcal{W}}$, then it holds for all admissible norms $\|\cdot\|_{\mathcal{W}}$. The space $k^2(I;\mathcal{W})$ does not have a unique positive inner product (only a family of equivalent inner Hilbert space inner products), but it does have a natural indefinite inner product, namely

$$\big[w_1(\cdot), w_2(\cdot)\big]_{k^2(I;\mathcal{W})} := \sum_{k\in I}\big[w_1(k), w_2(k)\big]_{\mathcal{W}}. \tag{2.3}$$

Because of (2.2), the sum above converges absolutely for all $w \in k^2(I;\mathcal{W})$. With this inner product $k^2(I;\mathcal{W})$ becomes a Kreĭn space, and each fundamental decomposition $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ induces a fundamental decomposition

$$k^2(I;\mathcal{W}) = -\ell^2(I;\mathcal{Y})\,[\dot{+}]\,\ell^2(I;\mathcal{U}), \tag{2.4}$$

where the norms in $\mathcal{Y}$ and $\mathcal{U}$ are the norms inherited from $-\mathcal{W}$ and $\mathcal{W}$, respectively, and $\ell^2(I;\mathcal{Y})$ and $\ell^2(I;\mathcal{U})$ stand for the standard Hilbert ℓ^2-spaces on the interval I: if $\mathcal{X}$ is a Hilbert space and I an discrete interval then $\ell^2(I;\mathcal{X})$ consists of all $\mathcal{X}$-valued sequences $x(\cdot)$ on I satisfying

$$\big\|x(\cdot)\big\|^2_{\ell^2(I;\mathcal{X})} := \sum_{k\in I}\big\|x(k)\big\|^2_{\mathcal{X}} < \infty. \tag{2.5}$$

In the sequel we abbreviate the cases where I is one of the intervals $\mathbb{Z}^-$, $\mathbb{Z}$, or $\mathbb{Z}^+$ as follows:

$$\begin{aligned} k^2_-(\mathcal{W}) &:= k^2\big(\mathbb{Z}^-;\mathcal{W}\big), & k^2(\mathcal{W}) &:= k^2(\mathbb{Z};\mathcal{W}), & k^2_+(\mathcal{W}) &:= k^2\big(\mathbb{Z}^+;\mathcal{W}\big),\\ \ell^2_-(\mathcal{X}) &:= \ell^2\big(\mathbb{Z}^-;\mathcal{X}\big), & \ell^2(\mathcal{X}) &:= \ell^2(\mathbb{Z};\mathcal{X}), & \ell^2_+(\mathcal{X}) &:= \ell^2\big(\mathbb{Z}^+;\mathcal{X}\big). \end{aligned}$$

If I and I' are two intervals with $I \subset I'$, then we frequently identify $k^2(I;\mathcal{W})$ with the subspace

$$\big\{w \in k^2(I';\mathcal{W}) \;\big|\; w(k) = 0 \text{ for } k \notin I\big\}$$

of $k^2(I';\mathcal{W})$, and in the same way we identify $\ell^2(I;\mathcal{X})$ with a subspace of $\ell^2(I';\mathcal{X})$.

As the following lemma shows, the condition $x \in \ell^\infty(I;\mathcal{X})$ in (2.1) is often redundant or almost redundant.

Lemma 2.1. *Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a passive s/s system, and let I be an discrete time interval, and let $(x(\cdot), w(\cdot))$ be a trajectory of Σ on I.*

(1) *If $I = [m,\infty)$ for some finite m, then $(x(\cdot), w(\cdot))$ is stable if and only if $w(\cdot) \in k^2(I;\mathcal{W})$.*
(2) *If I is unbounded to the left, then $(x(\cdot), w(\cdot))$ is stable if and only if $w(\cdot) \in k^2(I;\mathcal{W})$ and $\limsup_{m\to-\infty}\|x(m)\|_{\mathcal{X}} < \infty$.*

Proof. It follows from the nonnegativity of V that (1.8) holds. This implies both (1) and (2) since the sum in (1.8) stays bounded as $n \to \infty$ or $m \to -\infty$. $\square$

In the case of externally generated trajectories the preceding result simplifies as follows.

Lemma 2.2. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system, and let I be an discrete time interval, and let $(x(\cdot), w(\cdot))$ be an externally generated trajectory of Σ on I. Then $(x(\cdot), w(\cdot))$ is stable if and only if $w(\cdot) \in k^2(I; \mathcal{W})$. Moreover, if $I = [m, \infty)$ for some finite m, then*

$$\|x(n+1)\|^2_{\mathcal{X}} \leqslant [w(\cdot), w(\cdot)]_{k^2([m,n];\mathcal{W})}, \quad n \in I, \tag{2.6}$$

and if $I = (-\infty, k)$ (where we allow $k = \infty$), then

$$\|x(n+1)\|^2_{\mathcal{X}} \leqslant [w(\cdot), w(\cdot)]_{k^2((-\infty,n];\mathcal{W})}, \quad n \in I. \tag{2.7}$$

In particular, if $I = \mathbb{Z}^-$, then

$$\|x(0)\|^2_{\mathcal{X}} \leqslant [w(\cdot), w(\cdot)]_{k^2_-(\mathcal{W})}. \tag{2.8}$$

Proof. This follows from Lemma 2.1 and the definition of an externally generated trajectory. □

Formulas (1.8) and (2.6)–(2.8) explain why the Kreĭn spaces $k^2(I; \mathcal{W})$ appear naturally in connection with passive s/s systems.

In the sequel we shall need the following basic facts about stable trajectories of Σ.

Lemma 2.3. *The set of stable trajectories of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ have the following properties.*

(1) *Both the set of all stable trajectories and the set of all externally generated stable trajectories of Σ on some interval I (finite or infinite) are closed subspaces of $\ell^\infty(I; \mathcal{X}) \times \ell^2(I; \mathcal{W})$.*
(2) *If $(x(\cdot), w(\cdot))$ is a stable trajectory of Σ on some interval I and $n \in \mathbb{Z}$, then $(S^n x, S^n w)$ is a stable trajectory of Σ on $S^n I = \{k \in \mathbb{Z} \mid k - n \in I\}$, and $(x(\cdot), w(\cdot))$ is externally generated on I if and only if $(S^n x, S^n w)$ is externally generated on $S^n I$.*
(3) *The restriction of a stable trajectory on some interval I' to a subinterval $I \subset I'$ is a stable trajectory of Σ on I, and if I and I' have the same left end-point, then the restricted trajectory is externally generated if and only if the original trajectory is externally generated.*
(4) *If $(x(\cdot), w(\cdot))$ is an externally generated stable trajectory of Σ on an interval $I = [m, n)$ (where we allow $n = \infty$), and if we define $x(k) = 0$ and $w(k) = 0$ for $k < m$, then this extended pair of sequences is an externally generated stable trajectory of Σ on $(-\infty, n)$.*
(5) *Let $\mathcal{W} = -\mathcal{Y} [\dotplus] \mathcal{U}$ be a fundamental decomposition of $\mathcal{W}$. Then, for each $x_0 \in \mathcal{X}$ and each $u \in \ell^2_+(\mathcal{U})$ there exists a unique stable future trajectory $(x(\cdot), w(\cdot))$ of Σ satisfying $x(0) = x_0$ and $P_{\ell^2_+(\mathcal{U})} = u$.*
(6) *Every stable trajectory on some interval $I = (m, n]$ (where we allow $m = -\infty$) can be extended to a stable trajectory of Σ on (m, ∞).*
(7) *To each $\begin{bmatrix} x_1 \\ x_0 \\ w_0 \end{bmatrix} \in V$ there exists at least one stable future trajectory $(x(\cdot), w(\cdot))$ of Σ satisfying $x(0) = x_0$, $x(1) = x_1$, and $w(0) = w_0$.*

Proof. (1)–(4) Claim (1) follows from (1.3) and the fact that V is maximal nonnegative, and hence closed in the node space $\mathfrak{K}$. Properties (2)–(4) follow immediately from the definition of a stable trajectory.

(5) Let $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ be a fundamental decomposition. Then, by Theorem II.5.7, this decomposition is admissible for Σ, which means that for each $x_0 \in \mathcal{X}$ and $u \in \mathcal{U}^{\mathbb{Z}^+}$ the system Σ has a unique trajectory $(x(\cdot), w(\cdot))$ on $\mathbb{Z}^+$ satisfying $x(0) = x_0$ and $P_{\mathcal{U}^{\mathbb{Z}^+}} w(\cdot) = u(\cdot)$. For example, we may take $u \in \ell^2_+(\mathcal{U})$. It then follows from (1.8) that the corresponding trajectory is stable, since we have for all $n \in \mathbb{Z}^+$,

$$\begin{aligned}
&\|x(n+1)\|^2_{\mathcal{X}} - \sum_{k=0}^{n} [w(k), w(k)]_{\mathcal{W}} \\
&\quad = \|x(n+1)\|^2_{\mathcal{X}} + \sum_{k=0}^{n} \|P_{\mathcal{Y}} w(k)\|^2_{\mathcal{Y}} - \sum_{k=0}^{n} \|P_{\mathcal{U}} w(k)\|^2_{\mathcal{U}} \\
&\quad \leqslant \|x_0\|^2_{\mathcal{X}}.
\end{aligned} \tag{2.9}$$

(6) By property (2), we may without loss of generality suppose that $n = -1$. Let $(x'(\cdot), w'(\cdot))$ be the stable future trajectory of Σ given by (5) that satisfies $x'(0) = x(0)$ and $P_{\ell^2_+(\mathcal{U})} w(\cdot) = 0$. By defining $x(k) = x'(k)$ and $w(k) = w'(k)$ for $k > 0$ we get an trajectory on $I' = (m, \infty)$ whose restriction to $I = (m, -1]$ is the given trajectory of Σ.

(7) This is a special case of (6) with $I = \{0\}$. $\square$

Lemma 2.4. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system, and let $I = (-\infty, n)$ (where we allow $n = \infty$). Then the set of all compactly supported externally generated stable trajectories (i.e., trajectories $(x(\cdot), w(\cdot))$ that satisfy $x(k) = 0$ and $w(k) = 0$ for all k in some interval $(-\infty, m]$) is dense in the set of all externally generated stable trajectories of Σ on I in the topology inherited from $\ell^\infty(I; \mathcal{X}) \times k^2(I; \mathcal{W})$.*

Proof. Let $(x(\cdot), w(\cdot))$ be an externally generated stable trajectory of Σ on I, and let $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ be a fundamental decomposition of $\mathcal{W}$. By claims (2)–(5) of Lemma 2.3, for each $m \in I$ there is a unique externally generated stable trajectory $(x_m(\cdot), w_m(\cdot))$ of Σ on I satisfying $x(k) = 0$ and $w(k) = 0$ for $k \leqslant m$ and $P_{\ell^2(I;\mathcal{U})} w_m = P_{\ell^2([m,n);\mathcal{U})} w$. Define $x'_m(\cdot) = x(\cdot) - x_m(\cdot)$ and $w'_m(\cdot) = w(\cdot) - w_m(\cdot)$. Then $(x'_m(\cdot), w'(\cdot))$ is an externally generated trajectory of Σ on I, and by (2.7), for all $k \in I$,

$$\begin{aligned}
\|x'_m(k+1)\|^2_{\mathcal{X}} + \|P_{\ell^2((-\infty,k];\mathcal{Y})} w'_m\|^2 &\leqslant \|P_{\ell^2((-\infty,k];\mathcal{U})} w'_m\|^2 \\
&\leqslant \|P_{\ell^2((-\infty,m);\mathcal{U})} w\|^2.
\end{aligned}$$

This implies that

$$\|x'_m\|^2_{\ell^\infty((-\infty,n];\mathcal{X})} + \|P_{\ell^2((-\infty,n);\mathcal{Y})} w'_m\|^2 \leqslant 2\|P_{\ell^2((-\infty,m);\mathcal{U})} w\|^2,$$

where the right-hand side tends to zero as $m \to -\infty$. Thus, $x_m \to x$ in $\ell^\infty((-\infty, n]; \mathcal{X})$ and $w_m \to w$ in $k^2(I; \mathcal{W})$ as $m \to -\infty$. $\square$

Behaviors of passive state/signal systems

By *the* (*stable*) *behavior* induced by the passive s/s system Σ *on the interval* I we mean the set

$$\{w(\cdot) \mid (x(\cdot), w(\cdot)) \text{ is an externally generated stable trajectory of } \Sigma \text{ on } I\},$$

and we denote it by $\mathfrak{W}^{\Sigma}(I)$. Here we sometimes omit the upper index Σ if it is clear from the context which system this behavior is induced by. The cases where I is one of the intervals $\mathbb{Z}^-$, $\mathbb{Z}$, and $\mathbb{Z}^+$ are especially important, and we refer to these behaviors as the *past behavior* $\mathfrak{W}^{\Sigma}_{\text{past}}$, the *full behavior* $\mathfrak{W}^{\Sigma}_{\text{full}}$, and the *future behavior* $\mathfrak{W}^{\Sigma}_{\text{fut}}$ induced by the passive system Σ. Thus,

$$\mathfrak{W}^{\Sigma}_{\text{past}} = \mathfrak{W}^{\Sigma}(\mathbb{Z}^-), \qquad \mathfrak{W}^{\Sigma}_{\text{full}} = \mathfrak{W}^{\Sigma}(\mathbb{Z}), \qquad \mathfrak{W}^{\Sigma}_{\text{fut}} = \mathfrak{W}^{\Sigma}(\mathbb{Z}^+).$$

The following result is immediate.

Lemma 2.5. *To each* $w \in \mathfrak{W}^{\Sigma}_{\text{fut}}$ *there exists a unique* $x \in \ell^{\infty}_{+}(\mathcal{X})$ *such that* $(x(\cdot), w(\cdot))$ *is an externally generated stable trajectory of* Σ *on* $\mathbb{Z}^+$. *The same statement remains true if we replace* $\mathfrak{W}^{\Sigma}_{\text{fut}}$ *by* $\mathfrak{W}^{\Sigma}_{\text{full}}$ *or by* $\mathfrak{W}^{\Sigma}_{\text{past}}$ *and at the same time replace* $\mathbb{Z}^+$ *by* $\mathbb{Z}$ *or* $\mathbb{Z}^-$, *respectively.*

Proof. This follows from the definitions of $\mathfrak{W}^{\Sigma}_{\text{fut}}$, $\mathfrak{W}^{\Sigma}_{\text{full}}$, and $\mathfrak{W}^{\Sigma}_{\text{past}}$ and Lemma 2.2. □

The right-shift operators on $k^2_-(\mathcal{W})$, $k^2(\mathcal{W})$, and $k^2_+(\mathcal{W})$, are denoted by S_-, S, and S_+, respectively. The operator S_- is a co-isometry on $k^2_-(\mathcal{W})$, the operator S is unitary on $k^2(\mathcal{W})$, and the operator S_+ is an isometry on $k^2_+(\mathcal{W})$. The operators S_- and S_+ can be expressed in terms of the operator S by

$$S_- = \pi_- S|_{k^2_-(\mathcal{W})}, \qquad S_+ = S|_{k^2_+(\mathcal{W})},$$

where π_- is the orthogonal projection of $k^2(\mathcal{W})$ onto $k^2_-(\mathcal{W})$.

It will be shown in Theorem 2.8 below that the full behavior $\mathfrak{W}^{\Sigma}_{\text{full}}$ of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is a maximal S-reducing subspace of $k^2(\mathcal{W})$ (i.e., it is invariant under both S and S^{-1}). However, the converse is not true: $\mathfrak{W}^{\Sigma}_{\text{fut}}$ has one extra property, called *causality*, which is not a consequence of the fact that $\mathfrak{W}^{\Sigma}_{\text{full}}$ is maximal nonnegative and S-reducing. Let $\mathfrak{W}$ be a maximal nonnegative subspace of $k^2(\mathcal{W})$, and let $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ be a fundamental decomposition of $\mathcal{W}$. Then $k^2(\mathcal{W}) = -\ell^2(\mathcal{Y})\,[\dot{+}]\,\ell^2(\mathcal{U})$ is a fundamental decomposition of $\mathfrak{W}$. It follows from (2.8) that

$$\|x(0)\|^2_{\mathcal{X}} \leqslant -\|P_{\ell^2_-(\mathcal{Y})} w\|^2_{\ell^2_-(\mathcal{Y})} + \|P_{\ell^2_-(\mathcal{U})} w\|^2_{\ell^2_-(\mathcal{U})}.$$

In particular, if $\|P_{\ell^2_-(\mathcal{Y})} w\|^2_{\ell^2_-(\mathcal{U})} = 0$, then $\pi_- w = 0$.

Definition 2.6. A maximal nonnegative S-reducing subspace $\mathfrak{W}$ of $k^2(\mathcal{W})$ is causal if it is true for some fundamental decomposition $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ of $\mathcal{W}$ that

$$w(\cdot) \in \mathfrak{W} \quad \text{and} \quad P_{\ell^2_-(\mathcal{U})} w = 0 \quad \Rightarrow \quad \pi_- w(\cdot) = 0. \tag{2.10}$$

We shall see later that the choice of the fundamental decomposition $\mathcal{W} = -\mathcal{Y} [\dot{+}] \mathcal{U}$ in Definition 2.6 is not important: if (2.10) holds for *one* fundamental decomposition, then it holds for *every* fundamental decomposition of $\mathcal{W}$.

Not every maximal nonnegative S-reducing subspace of $k^2(\mathcal{W})$ is causal, as the following counter-example shows.

Example 2.7. Let $\mathcal{U}$ be a Hilbert space, and let $\mathcal{X}$ be the Kreĭn space $\mathcal{X} = -\mathcal{Y} [\dot{+}] \mathcal{U}$ where $\mathcal{Y} = \mathcal{U}$. Then $k^2(\mathcal{W}) = -\ell^2(\mathcal{Y}) [\dot{+}] \ell^2(\mathcal{U})$. Let

$$\mathfrak{W} = \left\{ \begin{bmatrix} S_{\mathcal{U}}^{-1} u \\ u \end{bmatrix} \;\middle|\; u \in \ell^2(\mathcal{U}) \right\}, \tag{2.11}$$

where $S_{\mathcal{U}}$ is the right-shift in $\ell^2(\mathcal{U})$. It is easy to see that $\mathfrak{W}^{[\perp]} = \mathfrak{W}$, i.e., $\mathfrak{W}$ is Lagrangian, hence maximal nonnegative (and also maximal nonpositive). It is also S-reducing. However, it is not causal: if $u \in \ell^2_+(\mathcal{U})$ and $u(0) \neq 0$, then $\left[\begin{smallmatrix} (S_{\mathcal{U}}^{-1} u)(-1) \\ u(-1) \end{smallmatrix}\right] = \left[\begin{smallmatrix} u(0) \\ 0 \end{smallmatrix}\right]$, so condition (2.10) does not hold.

Theorem 2.8. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system. Then the behaviors induced by Σ have the following properties.*

(1) $\mathfrak{W}^{\Sigma}_{\mathrm{fut}}$ *is a maximal nonnegative S_+-invariant subspace of $k^2_+(\mathcal{W})$.*
(2) $\mathfrak{W}^{\Sigma}_{\mathrm{full}}$ *is a maximal nonnegative S-reducing causal subspace of $k^2(\mathcal{W})$.*
(3) $\mathfrak{W}^{\Sigma}_{\mathrm{past}}$ *is a maximal nonnegative S_--invariant subspace of $k^2_-(\mathcal{W})$.*
(4) $\mathfrak{W}^{\Sigma}_{\mathrm{fut}} = \mathfrak{W}^{\Sigma}_{\mathrm{full}} \cap k_+(\mathcal{W})$.
(5) $\mathfrak{W}^{\Sigma}_{\mathrm{full}} = \bigvee_{n \in \mathbb{Z}^+} S^{-n} \mathfrak{W}^{\Sigma}_{\mathrm{fut}}$.
(6) $\mathfrak{W}^{\Sigma}_{\mathrm{past}} = \pi_- \mathfrak{W}^{\Sigma}_{\mathrm{full}}$.
(7) $\mathfrak{W}^{\Sigma}_{\mathrm{full}} = \bigcap_{n \in \mathbb{Z}^+} \{ w(\cdot) \in k^2(\mathcal{W}) \mid \pi_- S^{-n} w \in \mathfrak{W}^{\Sigma}_{\mathrm{past}} \}$.

Proof. *Step* 1: *Proofs of* (4), (6), *and* (7). These identities follow from Lemma 2.3.

Step 2: *Proof of* (1). The nonnegativity of $\mathfrak{W}^{\Sigma}_{\mathrm{fut}}$ follows from (2.6), and the S_+-invariance of $\mathfrak{W}^{\Sigma}_{\mathrm{fut}}$ follows from Lemma 2.3. It remains to prove that $\mathfrak{W}^{\Sigma}_{\mathrm{fut}}$ is *maximal* nonnegative in $k^2_+(\mathcal{W})$.

By definition, $w(\cdot) \in \mathfrak{W}^{\Sigma}_{\mathrm{fut}}$ if and only if there exists (a unique) bounded sequence $x(\cdot)$ such that $(x(\cdot), w(\cdot))$ is an externally generated stable trajectory of Σ on $\mathbb{Z}^+$. Let $\mathcal{W} = -\mathcal{Y} [\dot{+}] \mathcal{U}$ be a fundamental decomposition of $\mathcal{W}$. Then (2.4) with $I = \mathbb{Z}^+$ is a fundamental decomposition of $k^2_+(\mathcal{W})$, and by (2.9) with $n = 0$ and $x(0) = 0$,

$$\| P_{\mathcal{Y}} w(\cdot) \|_{\ell^2_+(\mathcal{Y})} \leqslant \| P_{\mathcal{U}} w(\cdot) \|_{\ell^2_+(\mathcal{U})}.$$

By part (5) of Lemma 2.3, the function $P_{\mathcal{U}} w(\cdot)$ can be an arbitrary function in $\ell^2_+(\mathcal{U})$. This implies that there exists a bounded linear operator $\mathfrak{D}_+$ such that

$$\mathfrak{W}^{\Sigma}_{\mathrm{fut}} = \left\{ \begin{bmatrix} \mathfrak{D}_+ u \\ u \end{bmatrix} \;\middle|\; u \in \ell^2_+(\mathcal{U}) \right\}. \tag{2.12}$$

Thus, $\mathfrak{W}^{\Sigma}_{\mathrm{fut}}$ is the graph of a contraction $\mathfrak{D}_+ : \ell^2_+(\mathcal{U}) \to \ell^2_+(\mathcal{Y})$ and hence maximal nonnegative.

Step 3: $\mathfrak{W}^{\Sigma}_{\mathrm{full}}$ *is closed in* $k^2(\mathcal{W})$. Let $w_j(\cdot)$ be a sequence in $\mathfrak{W}^{\Sigma}_{\mathrm{full}}$ converging to some $w \in k^2(\mathcal{W})$. Then, to each w_j there corresponds a sequence $x_j(\cdot) \in \ell^\infty(\mathcal{X})$ satisfying $x_j(n) \to 0$ as $n \to -\infty$ such that $(x_j(\cdot), w_j(\cdot))$ is an externally generated full stable trajectory of Σ. The sequence $w_j(\cdot)$ is a Cauchy sequence in $k^2(\mathcal{W})$, and it follows from (2.7) that $x_j(\cdot)$ is a Cauchy sequence in $\ell^\infty(\mathcal{X})$. Thus, $x_j(\cdot)$ tends to a limit $x(\cdot)$ in $\ell^\infty(\mathcal{X})$ satisfying $x(n) \to 0$ as $n \to -\infty$. The generating subspace V is closed, and it follows from (1.3) that $(x(\cdot), w(\cdot))$ is an externally generated stable trajectory of Σ on $\mathbb{Z}$. Thus, $w \in \mathfrak{W}^{\Sigma}_{\mathrm{full}}$, and this proves that $\mathfrak{W}^{\Sigma}_{\mathrm{full}}$ is closed.

Step 4: *Proofs of* (2) *and* (5). The nonnegativity of $\mathfrak{W}^{\Sigma}_{\mathrm{full}}$ follows from (2.7), and that $\mathfrak{W}^{\Sigma}_{\mathrm{full}}$ is S-reducing follows from Lemma 2.3.

Recall that $\mathfrak{W}^{\Sigma}_{\mathrm{fut}}$ has the graph representation (2.12) for some contraction $\mathfrak{D}_+ : \ell^2_+(\mathcal{U}) \to \ell^2_+(\mathcal{Y})$, where $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ is a fundamental decomposition of $\mathcal{W}$. The S_+-invariance of $\mathfrak{W}^{\Sigma}_{\mathrm{fut}}$ implies that $\mathfrak{D}_+$ is shift-invariant in the sense that $\mathfrak{D}_+ S_+ = S_+ \mathfrak{D}_+$. Let $\ell^2_0(\mathcal{U})$ be the subset of $\ell^2(\mathcal{U})$ consisting of those sequences in $\ell^2(\mathcal{U})$ whose support is bounded to the left. It is possible to define a contraction $\mathfrak{D} : \ell^2_0(\mathcal{U}) \to \ell^2(\mathcal{Y})$ in the following way: If $u \in \ell^2_0(\mathcal{U})$ vanishes on $(-\infty, n]$, then we define $\mathfrak{D}u = S^{-m}\mathfrak{D}_+ S^m u$, where m is chosen to be so large that $S^m u$ vanishes on $\mathbb{Z}^-$. The result is independent of the particular value of m because $\mathfrak{D}_+ S_+ = S_+ \mathfrak{D}_+$. Since $\ell^2_0(\mathcal{U})$ is dense in $\ell^2(\mathcal{U})$ we can extend $\mathfrak{D}$ to a contraction $\ell^2(\mathcal{U}) \to \ell^2(\mathcal{Y})$. This contraction is causal in the sense that $\mathfrak{D}\ell^2_+(\mathcal{U}) \subset \ell^2_+(\mathcal{Y})$, and it is shift-invariant in the sense that $\mathfrak{D}Su = S\mathfrak{D}u$ for all $u \in \ell^2(\mathcal{U})$. Moreover, $\mathfrak{D}_+ = \mathfrak{D}|_{\ell^2(\mathcal{W})}$.

It follows from (2.12) with $\mathfrak{D}_+ = \mathfrak{D}|_{\ell^2(\mathcal{W})}$ that

$$S^{-n}\mathfrak{W}^{\Sigma}_{\mathrm{fut}} = \left\{ \begin{bmatrix} S^{-n}_{\mathcal{Y}}\mathfrak{D}_+ u_+ \\ S^{-n}_{\mathcal{U}} u_+ \end{bmatrix} \,\middle|\, u_+ \in \ell^2_+(\mathcal{U}) \right\} = \left\{ \begin{bmatrix} \mathfrak{D} S^{-n}_{\mathcal{Y}} u_+ \\ S^{-n}_{\mathcal{U}} u_+ \end{bmatrix} \,\middle|\, u_+ \in \ell^2_+(\mathcal{U}) \right\},$$

where $\bigvee_{n\in\mathbb{Z}^+} \ell^2_+(\mathcal{U}) = \ell^2(\mathcal{U})$. Thus, $\bigvee_{z\in\mathbb{Z}^+} S^{-n}\mathfrak{W}^{\Sigma}_{\mathrm{fut}} = \{\left[\begin{smallmatrix} \mathfrak{D}u \\ u \end{smallmatrix}\right] \mid u \in \ell^2(\mathcal{U})\}$. This graph representation implies that $\bigvee_{z\in\mathbb{Z}^+} S^{-n}\mathfrak{W}^{\Sigma}_{\mathrm{fut}}$ is maximal nonnegative in $k^2(\mathcal{W})$.

It follows from Lemma 2.3 that $\bigcup_{z\in\mathbb{Z}^+} S^{-n}\mathfrak{W}^{\Sigma}_{\mathrm{fut}} \subset \mathfrak{W}^{\Sigma}_{\mathrm{full}}$, and since $\mathfrak{W}^{\Sigma}_{\mathrm{full}}$ is closed, we have $\bigvee_{z\in\mathbb{Z}^+} S^{-n}\mathfrak{W}^{\Sigma}_{\mathrm{fut}} \subset \mathfrak{W}^{\Sigma}_{\mathrm{full}}$. Here $\bigvee_{z\in\mathbb{Z}^+} S^{-n}\mathfrak{W}^{\Sigma}_{\mathrm{fut}}$ is maximal nonnegative, and $\mathfrak{W}^{\Sigma}_{\mathrm{full}}$ is nonnegative. Thus, $\bigvee_{z\in\mathbb{Z}^+} S^{-n}\mathfrak{W}^{\Sigma}_{\mathrm{fut}} = \mathfrak{W}^{\Sigma}_{\mathrm{full}}$, and hence $\mathfrak{W}^{\Sigma}_{\mathrm{full}}$ is maximal nonnegative and (5) holds. In particular,

$$\mathfrak{W}^{\Sigma}_{\mathrm{full}} = \left\{ \begin{bmatrix} \mathfrak{D}u \\ u \end{bmatrix} \,\middle|\, u \in \ell^2(\mathcal{U}) \right\}. \tag{2.13}$$

That $\mathfrak{W}^{\Sigma}_{\mathrm{full}}$ is causal follows from this graph representation and the fact that $\mathfrak{D}\ell^2_+(\mathcal{U}) \subset \ell^2_+(\mathcal{Y})$.

Step 5: *Proofs of* (3). That $\mathfrak{W}^{\Sigma}_{\mathrm{past}}$ is S_--invariant follows from Lemma 2.3. The graph representation (2.13) together with (6) and the fact that $\mathfrak{D}\ell^2_+(\mathcal{U}) \subset \ell^2_+(\mathcal{Y})$ implies that $\mathfrak{W}^{\Sigma}_{\mathrm{past}}$ has the graph representation

$$\mathfrak{W}^{\Sigma}_{\mathrm{past}} = \left\{ \begin{bmatrix} \pi_-\mathfrak{D}_- u \\ \pi_- u \end{bmatrix} \,\middle|\, u \in \ell^2(\mathcal{U}) \right\} = \left\{ \begin{bmatrix} \mathfrak{D}_- u \\ u \end{bmatrix} \,\middle|\, u \in \ell^2_-(\mathcal{U}) \right\}, \tag{2.14}$$

where $\mathfrak{D}_- := \pi_-\mathfrak{D}|_{\ell^2_-(\mathcal{U})}$ is a contraction $\ell^2_-(\mathcal{U}) \to \ell^2_-(\mathcal{Y})$. This graph representation implies that $\mathfrak{W}^\Sigma_{\rm past}$ is maximal nonnegative in $k^2_-(\mathcal{W})$. $\square$

Corollary 2.9. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system. Then each one of the past, full, and future stable behaviors $\mathfrak{W}^\Sigma_{\rm past}$, $\mathfrak{W}^\Sigma_{\rm full}$, and $\mathfrak{W}^\Sigma_{\rm fut}$ of Σ determines the other two uniquely.*

Proof. This follows from claims (4)–(7) in Theorem 2.8. $\square$

Passive future, full, and past behaviors

Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system. According to Theorem 2.8, the future behavior $\mathfrak{W}^\Sigma_{\rm fut}$ of Σ is a maximal nonnegative S_+-invariant subspace of $k^2_+(\mathcal{W})$, the full behavior $\mathfrak{W}^\Sigma_{\rm fut}$ of Σ is a maximal nonnegative S-reducing causal subspace of $k^2(\mathcal{W})$, and the past behavior $\mathfrak{W}^\Sigma_{\rm past}$ of Σ is a maximal nonnegative S_--invariant subspace of $k^2_-(\mathcal{W})$. It will be shown in Section 7 that every maximal nonnegative S_+-invariant subspace of $k^2_+(\mathcal{W})$ is the future behavior of a passive s/s system, and analogously, it will be shown in Section 8 that every maximal nonnegative S_--invariant subspace of $k^2_-(\mathcal{W})$ is the past behavior of a passive s/s system. We shall also see that every maximal nonnegative S-reducing causal subspace of $k^2(\mathcal{W})$ is the full behavior of a passive s/s system. In view of these three facts the following definitions are natural.

Definition 2.10. Let $\mathcal{W}$ be a Kreĭn space.

(1) A maximal nonnegative S_+-invariant subspace of $k^2_+(\mathcal{W})$ is called a *passive future behavior* on the Kreĭn (signal) space $\mathcal{W}$.
(2) A maximal nonnegative S-reducing *causal* subspace of $k^2(\mathcal{W})$ is called a *passive full behavior* on the (signal) space $\mathcal{W}$.
(3) A maximal nonnegative S_--invariant subspace of $k^2_-(\mathcal{W})$ is called a *passive past behavior* on the (signal) space $\mathcal{W}$.

The basic connections between passive future, full, and past behaviors are described in the following theorem.

Theorem 2.11. *Let $\mathcal{W}$ be a Kreĭn space.*

(1) *If $\mathfrak{W}$ is a passive full behavior on $\mathfrak{W}$, and if we define $\mathfrak{W}_+$ and $\mathfrak{W}_-$ by*

$$\mathfrak{W}_+ := \mathfrak{W} \cap k^2_+(\mathcal{W}), \qquad \mathfrak{W}_- := \pi_-\mathfrak{W}, \tag{2.15}$$

then $\mathfrak{W}_+$ and $\mathfrak{W}_-$ are passive future and past behaviors on $\mathcal{W}$, respectively, and $\mathfrak{W}$ can be recovered from $\mathfrak{W}_+$ and from $\mathfrak{W}_-$ by the formulas

$$\mathfrak{W} = \bigvee_{n\in\mathbb{Z}^+} S^{-n}\mathfrak{W}_+, \tag{2.16}$$

$$\mathfrak{W} = \bigcap_{n\in\mathbb{Z}^+} \bigl\{ w(\cdot) \in k^2(\mathcal{W}) \bigm| \pi_- S^{-n} w \in \mathfrak{W}_- \bigr\}. \tag{2.17}$$

(2) *If* $\mathfrak{W}_+$ *is a passive future behavior on* $\mathcal{W}$*, and if we define* $\mathfrak{W}$ *by* (2.16)*, then* $\mathfrak{W}$ *is a passive full behavior on* $\mathcal{W}$ *and* $\mathfrak{W}_+ = \mathfrak{W} \cap k^2_+(\mathcal{W})$.

(3) *If* $\mathfrak{W}_-$ *is a passive past behavior on* $\mathcal{W}$*, and if we define* $\mathfrak{W}$ *by* (2.17)*, then* $\mathfrak{W}$ *is a passive full behavior on* $\mathcal{W}$ *and* $\mathfrak{W}_- = \pi_-\mathfrak{W}$.

Proof. Most of the proof of this theorem is very similar to the proof of Theorem 2.8, but some of the details are different.

(1) Let $\mathcal{W} = -\mathcal{Y} \, [\dotplus] \, \mathcal{U}$ be a fundamental decomposition of $\mathfrak{W}$. Then $k^2(\mathcal{W}) = -\ell^2(\mathcal{Y}) \, [\dotplus] \, \ell^2(\mathcal{U})$ is a fundamental decomposition of $k^2(\mathcal{W})$, and the maximal nonnegativity of $\mathfrak{W}$ implies that it has a graph representation

$$\mathfrak{W} = \left\{ \begin{bmatrix} \mathfrak{D}u \\ u \end{bmatrix} \,\middle|\, u \in \ell^2(\mathcal{U}) \right\} \tag{2.18}$$

for some contraction $\mathfrak{D} : \ell^2(\mathcal{U}) \to \ell^2(\mathcal{Y})$. Since $\mathfrak{W}$ is S-reducing, we have $S_{\mathcal{Y}}\mathfrak{D} = \mathfrak{D}S_{\mathcal{U}}$, and since $\mathfrak{W}$ is causal, $\mathfrak{D}\ell^2_+(\mathcal{U}) \subset \ell^2_+(\mathcal{Y})$. This, together with (2.15) implies that $\mathfrak{W}_\pm$ have the graph representations

$$\mathfrak{W}_+ = \left\{ \begin{bmatrix} \mathfrak{D}_+u \\ u \end{bmatrix} \,\middle|\, u \in \ell^2_+(\mathcal{U}) \right\}, \tag{2.19}$$

$$\mathfrak{W}_- = \left\{ \begin{bmatrix} \mathfrak{D}_-u \\ u \end{bmatrix} \,\middle|\, u \in \ell^2_-(\mathcal{U}) \right\}, \tag{2.20}$$

where $\mathfrak{D}_+ = \mathfrak{D}|_{\ell^2_+(\mathcal{U})}$ and $\mathfrak{D}_- = \pi_-\mathfrak{D}|_{\ell^2_-(\mathcal{U})}$ are contractions $\ell^2_\pm(\mathcal{U}) \to \ell^2_\pm(\mathcal{Y})$. These two graph representations with respect to the fundamental decompositions $k^2_\pm(\mathcal{W}) = -\ell^2_\pm(\mathcal{Y}) \, [\dotplus] \, \ell^2_\pm(\mathcal{U})$ imply that $\mathfrak{W}_\pm$ are maximal nonnegative in $k^2_\pm(\mathcal{W})$. That $\mathfrak{W}_+$ is S_+-invariant follows from its definition $\mathfrak{W}_+ = \mathfrak{W} \cap k^2_+(\mathcal{W})$ and the fact that $S\mathfrak{W} = S$. The S_--invariance of $\mathfrak{W}_-$ is proved by the following computation:

$$S_-\mathfrak{W}_- = \pi_- S \pi_- \mathfrak{W} = \pi_- \pi_{(-\infty,0]} S\mathfrak{W} = \pi_-\mathfrak{W} = \mathfrak{W}_-. \tag{2.21}$$

Thus, $\mathfrak{W}_+$ and $\mathfrak{W}_-$ are passive future and past behaviors, respectively.

A proof of the fact that $\bigvee_{z\in\mathbb{Z}^+} S^{-n}\mathfrak{W}_+$ is maximal nonnegative in $k^2(\mathcal{W})$ is contained in step 4 of the proof of Theorem 2.8 (with $\mathfrak{W}^\Sigma_{\text{fut}}$ replaced by $\mathfrak{W}_+$), and essentially the same proof shows that $\bigvee_{z\in\mathbb{Z}^+} S^{-n}\mathfrak{W}_+ = \mathfrak{W}$ (this time we have $\bigcup_{z\in\mathbb{Z}^+} S^{-n}\mathfrak{W}_+ \subset \mathfrak{W}$ since $\mathfrak{W}$ is S-reducing and $\mathfrak{W}_+ \subset \mathfrak{W}$).

Let $\mathfrak{W}_n := \{w(\cdot) \in k^2(\mathcal{W}) \mid \pi_- S^{-n} w \in \mathfrak{W}_-\}$, and let $\mathfrak{W}' := \bigcap_{n\in\mathbb{Z}^+} \mathfrak{W}_n$. The fact that $\mathfrak{W}$ is S-reducing and that $\pi_-\mathfrak{W} = \mathfrak{W}_-$ implies that $\mathfrak{W} \subset \mathfrak{W}'$. Each $\mathfrak{W}_n$ is nonnegative in $k^2((-\infty, n]; \mathcal{W})$ since $\mathfrak{W}_-$ is nonnegative in $k^2_-(\mathcal{W})$. For each $w \in \mathfrak{W}'$ we have $\pi_{(-\infty,n]} w(\cdot) \in \mathfrak{W}_n$, and hence

$$\big[w(\cdot), w(\cdot)\big]_{k^2(\mathcal{W})} = \lim_{n\to+\infty} \big[\pi_{(-\infty,n]} w(\cdot), \pi_{(-\infty,n]} w(\cdot)\big]_{k^2((-\infty,n];\mathcal{W})} \geqslant 0, \quad w \in \mathfrak{W}'.$$

Thus, $\mathfrak{W} \subset \mathfrak{W}'$ where $\mathfrak{W}$ is maximal nonnegative and $\mathfrak{W}'$ is nonnegative, and hence $\mathfrak{W} = \mathfrak{W}'$.

(2) Since $\mathfrak{W}_+$ is maximal nonnegative, it has a graph representation of the type (2.19) for some contraction $\mathfrak{D}_+ : \ell^2_+(\mathcal{U}) \to \ell^2_+(\mathcal{Y})$, where $\mathcal{W} = -\mathcal{Y}[\dot{+}]\mathcal{U}$ is a fundamental decomposition of $\mathcal{W}$. The same argument that we used in step 4 in the proof of Theorem 2.8 shows that $\bigvee_{z\in\mathbb{Z}^+} S^{-n}\mathfrak{W}_+$ is passive full behavior on $\mathcal{W}$, and that $\bigvee_{z\in\mathbb{Z}^+} S^{-n}\mathfrak{W}_+ = \mathfrak{W}$ whenever $\mathfrak{W}$ is a S-reducing closed nonnegative subspace of $k^2(\mathcal{W})$ satisfying $\mathfrak{W}_+ \subset \mathfrak{W}$.

(3) Since $\mathfrak{W}_-$ is maximal nonnegative, it has a graph representation of the type (2.20) for some contraction $\mathfrak{D}_- : \ell^2_-(\mathcal{U}) \to \ell^2_-(\mathcal{Y})$, where $\mathcal{W} = -\mathcal{Y}[\dot{+}]\mathcal{U}$ is a fundamental decomposition of $\mathcal{W}$. With the help of $\mathfrak{D}_-$ we can define a contraction $\mathfrak{D} : \ell^2(\mathcal{U}) \to \ell^2(\mathcal{Y})$ in the following way. We first define the sequence of contractions $\mathfrak{D}^n : \ell^2(\mathcal{U}) \to \ell^2(\mathcal{Y})$ by $\mathfrak{D}^n u = S^n\mathfrak{D}_-\pi_-S^{-n}u$, $n \geqslant 0$. The right-shift invariance of $\mathfrak{D}_-$ implies that, for all $m \geqslant n$,

$$\pi_{(-\infty,n]}\mathfrak{D}_m = S^n\pi_-S^{m-n}\mathfrak{D}_-\pi_-S^{-m} = S^n\mathfrak{D}_-\pi_-S^{m-n}\pi_-S^{-m} = \mathfrak{D}_n.$$

Thus, for each $u \in \ell^2(\mathcal{U})$ and all $m \in \mathbb{Z}^+$, $\|\mathfrak{D}_m u\|_{\ell^2(\mathcal{Y})} \leqslant \|u\|_{\ell^2(\mathcal{U})}$, and $\pi_{(-\infty,n]}\mathfrak{D}_m u$ is independent of m for $m \geqslant n$. This implies that $\mathfrak{D}_m u$ tends weakly to a limit $y \in \ell^2(\mathcal{Y})$.

Thus, for each $u \in \ell^2(\mathcal{U})$ and $m \geqslant n$,

$$\big\|(\mathfrak{D}_m - \mathfrak{D}_n)u\big\|_{\ell^2(\mathcal{Y})} \leqslant \big\|\pi_{(n,\infty)}(\mathfrak{D}_m - \mathfrak{D}_n)u\big\|_{\ell^2(\mathcal{Y})} \leqslant 2\|\pi_{(n,\infty)}u\|_{\ell^2(\mathcal{U})},$$

which tends to zero as $n \to +\infty$. Thus, $\mathfrak{D}_n$ tends strongly to a limit contraction $\mathfrak{D} : \ell^2(\mathcal{U}) \to \ell^2(\mathcal{Y})$. This contraction is causal in the sense that $\mathfrak{D}\ell^2_+(\mathcal{U}) \subset \ell^2_+(\mathcal{Y})$, and it is shift-invariant in the sense that $\mathfrak{D}Su = S\mathfrak{D}u$ for all $u \in \ell^2(\mathcal{U})$.

Define $\mathfrak{W}$ by (2.18). Then, by construction, $\mathfrak{D}S = S\mathfrak{D}$, $\mathfrak{D}\ell^2_+(\mathcal{U}) \subset \ell^2_+(\mathcal{Y})$, and $\mathfrak{D}_- = \pi_-\mathfrak{D}|_{\ell^2_-(\mathcal{U})}$. This implies that $\mathfrak{W}$ is a passive full behavior on $\mathcal{W}$ satisfying $\mathfrak{W}_- = \pi_-\mathfrak{W}$. That formula (2.17) holds follows from claim (1). □

Lemma 2.12. *Let $\mathfrak{W}_-$ be a passive past behavior on a Kreĭn space $\mathcal{W}$. Then the set of all $w(\cdot) \in \mathfrak{W}_-$ with finite support (i.e., $w(k) = 0$ for all k in some interval $(-\infty, n]$) is a dense subspace of $\mathfrak{W}_-$.*

Proof. By (2.15) and (2.16),

$$\mathfrak{W}_- = \pi_-\mathfrak{W} = \pi_- \bigvee_{n\in\mathbb{Z}^+} S^{-n}\big(\mathfrak{W} \cap k^2_+(\mathcal{W})\big) = \bigvee_{n\in\mathbb{Z}^+} \pi_-S^{-n}\big(\mathfrak{W} \cap k^2_+(\mathcal{W})\big),$$

where each sequence in $\pi_-S^{-n}(\mathfrak{W} \cap k^2_+(\mathcal{W}))$ has finite support. □

Remark 2.13. By Theorem 2.11, the map $\mathfrak{W} \mapsto \mathfrak{W} \cap k^2_+(\mathcal{W})$ is a bijection from the set of all passive full behaviors on $\mathcal{W}$ onto the set of all passive future behaviors on $\mathcal{W}$, with inverse $\mathfrak{W}_+ \mapsto \bigvee_{n\in\mathbb{Z}^+} S^{-n}\mathfrak{W}_+$. Likewise, the map $\mathfrak{W} \mapsto \pi_-\mathfrak{W}$ is a bijection from the set of all passive full behaviors on $\mathcal{W}$ onto the set of all passive past behaviors on $\mathcal{W}$, with inverse $\mathfrak{W}_- \mapsto \bigcap_{n\in\mathbb{Z}^+}\{w(\cdot) \in k^2(\mathcal{W}) \mid \pi_-S^{-n}w \in \mathfrak{W}_-\}$. Thus, formulas (2.15)–(2.17) define one-to-one correspondences between a passive future behavior $\mathfrak{W}_+$, a passive full behavior $\mathfrak{W}$, and a passive past behavior $\mathfrak{W}_-$: any one of these can be used to define the two others.

Let us go back to Example 2.7.

Example 2.14. Let $\mathfrak{W}$ be the Lagrangian subspace of $k^2(-\mathcal{Y}\,[\dotplus]\,\mathcal{U})$ defined in (2.11). As we saw in Example 2.7, $\mathfrak{W}$ is not causal. Define $\mathfrak{W}_\pm$ by (2.15). Then

$$\mathfrak{W}_+ = \left\{ \begin{bmatrix} S^{-1}u \\ u \end{bmatrix} \;\middle|\; u(\cdot) \in \ell_+^2(\mathcal{U}) \text{ with } u(k) = 0 \text{ for all } k \leqslant 0 \right\}, \tag{2.22}$$

$$\mathfrak{W}_- = \left\{ \begin{bmatrix} y \\ S_- y \end{bmatrix} \;\middle|\; y(\cdot) \in \ell^2(\mathcal{Y}) \right\}. \tag{2.23}$$

The subspace $\mathfrak{W}_+$ is not maximal nonnegative since the projection onto the positive component in the fundamental decomposition $k_+^2(\mathcal{W}) = -\ell_+^2(\mathcal{Y})\,[\dotplus]\,\ell_+^2(\mathcal{U})$ is not all of $\ell_+^2(\mathcal{U})$, and the subspace $\mathfrak{W}_-$ is not even nonnegative: if $u \in \ell_+^2(\mathcal{U})$ with $u(0) \neq 0$, then $\pi_-\left[\begin{smallmatrix} S^{-1}u \\ u \end{smallmatrix}\right] \in \mathfrak{W}_-$ and

$$\left[\pi_- \begin{bmatrix} S^{-1}u \\ u \end{bmatrix}, \pi_- \begin{bmatrix} S^{-1}u \\ u \end{bmatrix}\right]_{k_-^2(\mathcal{W})} = -\left\|u(0)\right\|_{\mathcal{U}}^2 < 0.$$

Remark 2.15. Our proof of claim (2) in Theorem 2.11 shows that a stronger statement is true than the one recorded in the theorem: *If $\mathfrak{W}$ is a closed nonnegative S-reducing subspace of $k^2(\mathcal{W})$ which contains a maximal nonnegative S_+-invariant subspace $\mathfrak{W}_+$ of $k_+^2(\mathcal{W})$, then $\mathfrak{W}$ is given by* (2.16). Thus, $\mathfrak{W}$ is uniquely determined by $\mathfrak{W}_+$ within the class of all closed nonnegative S-reducing subspaces of $k^2(\mathcal{W})$, and not only within the class of all maximal nonnegative causal S-reducing subspaces of $k^2(\mathcal{W})$. A similar extension of claim (3) is also valid, as explained in Remark 3.10 below.

3. Anti-passive reflected systems and behaviors

Since the generating subspace V of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is maximal nonnegative, its orthogonal companion $V^{[\perp]}$ is maximal nonpositive, and it generates an *anti-passive reflected state/signal system* $\Sigma^\dagger = (V^{[\perp]}; \mathcal{X}, \mathcal{W})$. The trajectories $(x^\dagger(\cdot), w^\dagger(\cdot))$ of $\Sigma^\dagger$ satisfy

$$\begin{bmatrix} x^\dagger(n+1) \\ x^\dagger(n) \\ w^\dagger(n) \end{bmatrix} \in V^{[\perp]}, \quad n \in I. \tag{3.1}$$

It differs from a standard passive s/s system in the sense that trajectories always can be continued *backward in time* instead of forward in time, and it is not a special case of a state/signal system in the sense of [2–5]. If we define V_* by (1.4), then V_* is maximal nonnegative in the Kreĭn space $-\mathcal{X}\,[\dotplus]\,\mathcal{X}\,[\dotplus]\,-\mathcal{W}$, and it generates a standard passive s/s system $\Sigma_* = (V_*; \mathcal{X}, -\mathcal{W})$, which we in [3–5] called the *adjoint* of the s/s system Σ. Here we shall instead refer to Σ_* as the *passive dual* of Σ, and call $\Sigma^\dagger$ the *anti-passive dual* of Σ. The trajectories of Σ_* and $\Sigma^\dagger$ differ from each other by a time reflection, and, in addition, their signal spaces also differ from each other (the signal space of $\Sigma^\dagger$ is $\mathcal{W}$ and the signal space of Σ_* is $-\mathcal{W}$). Because of the indexing conventions used in (1.3) and (3.1), the reflection in the state component $x(\cdot)$ differs slightly from the reflection in the signal component $w(\cdot)$: $(x(\cdot), w(\cdot))$ is a trajectory of Σ_* on an interval I if and only if the function $(x^\dagger(\cdot), w^\dagger(\cdot))$ defined by $x^\dagger(n) = x(-n)$ and $w^\dagger(n) = w(-n-1)$ is a trajectory of $\Sigma^\dagger$ on $I^\dagger = \{z \in \mathbb{Z} \mid -z-1 \in I\}$.

Stable trajectories of an anti-passive reflected s/s system are defined in the same way as for a passive s/s system, and we still refer to trajectories defined on $\mathbb{Z}^-$, $\mathbb{Z}$, and $\mathbb{Z}^+$ as past, full, and future trajectories. Past, full, and future trajectories are also defined in the same way as for passive s/s systems, i.e., "past" always refers to the time interval $\mathbb{Z}^-$, "full" to the time interval $\mathbb{Z}$, and "future" to the time interval $\mathbb{Z}^+$. However, since the natural direction of evolution of an anti-passive reflected s/s system is opposite to the natural direction of evolution of a passive system, a trajectory $(x^\dagger(\cdot), w^\dagger(\cdot))$ of an anti-passive reflected s/s system is *externally generated* if the state vanishes at the *right end-point* of the interval of definition, i.e., $x^\dagger(n) = 0$ when $I = (m, n)$ and $\lim_{n\to+\infty} x(n) = 0$ when $I = (m, \infty)$.

Lemma 3.1. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system, and let $\Sigma^\dagger = (V^{[\perp]}; \mathcal{X}, \mathcal{W})$ be its anti-passive dual.*

(1) *Σ is forward conservative if and only if every trajectory of Σ on every interval I is also a trajectory of $\Sigma^\dagger$ on I.*
(2) *Σ is backward conservative if and only if every trajectory of $\Sigma^\dagger$ on every interval I is also a trajectory of Σ on I.*
(3) *Σ is conservative if and only if Σ and $\Sigma^\dagger$ have the same set of trajectories on every interval I.*

Proof. This is true, because, by definition, Σ is forward conservative if and only if $V \subset V^{[\perp]}$, Σ is backward conservative if and only if $V^{[\perp]} \subset V$, and Σ is conservative if and only if $V = V^{[\perp]}$. □

The trajectories of the original passive s/s system Σ are "orthogonal" to trajectories of the anti-passive dual system $\Sigma^\dagger$ in the following sense:

Lemma 3.2. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system, and let $\Sigma^\dagger = (V^{[\perp]}; \mathcal{X}, \mathcal{W})$ be the anti-passive dual of Σ. Let I be a subinterval of $\mathbb{Z}$, let $(x(\cdot), w(\cdot))$ be a stable trajectory of Σ on I, and let $(x^\dagger(\cdot), w^\dagger(\cdot))$ be a stable trajectory of $\Sigma^\dagger$ on I.*

(1) *If $I = [m, n)$ for some finite $n > m$, then*

$$\big(x(n), x^\dagger(n)\big)_{\mathcal{X}} = \big(x(m), x^\dagger(m)\big)_{\mathcal{X}} + \big[w(\cdot), w^\dagger(\cdot)\big]_{k^2(I;\mathcal{W})}. \tag{3.2}$$

(2) *If $I = (-\infty, n)$ for some finite n, then $\lim_{m\to-\infty}(x(m), x^\dagger(m))_{\mathcal{X}}$ exists, and*

$$\big(x(n), x^\dagger(n)\big)_{\mathcal{X}} = \lim_{m\to-\infty}\big(x(m), x^\dagger(m)\big)_{\mathcal{X}} + \big[w(\cdot), w^\dagger(\cdot)\big]_{k^2(I;\mathcal{W})}. \tag{3.3}$$

(3) *If $I = [m, \infty)$ for some finite m, then $\lim_{n\to+\infty}(x(n), x^\dagger(n))_{\mathcal{X}}$ exists, and*

$$\lim_{n\to+\infty}\big(x(n), x^\dagger(n)\big)_{\mathcal{X}} = \big(x(m), x^\dagger(m)\big)_{\mathcal{X}} + \big[w(\cdot), w^\dagger(\cdot)\big]_{k^2(I;\mathcal{W})}. \tag{3.4}$$

Proof. This follows immediately from (1.3) and (3.1). □

By *the* (*stable*) *behavior* induced by the anti-passive s/s system $\Sigma^\dagger$ *on the interval* I we mean the set

$$\bigl\{w^\dagger(\cdot) \bigm| \bigl(x^\dagger(\cdot), w^\dagger(\cdot)\bigr) \text{ is an externally generated stable trajectory of } \Sigma^\dagger \text{ on } I\bigr\},$$

and we denote it by $\mathfrak{W}^{\Sigma^\dagger}(I)$. We refer to the behaviors on the intervals $\mathbb{Z}^-$, $\mathbb{Z}$, and $\mathbb{Z}^+$ as the *past behavior* $\mathfrak{W}^{\Sigma^\dagger}_{\mathrm{past}}$, the *full behavior* $\mathfrak{W}^{\Sigma^\dagger}_{\mathrm{full}}$, and the *future behavior* $\mathfrak{W}^{\Sigma^\dagger}_{\mathrm{fut}}$ induced by the anti-passive system $\Sigma^\dagger$.

In the next theorem we need the notion of an anti-causal maximal nonpositive S-reducing subspace of $k^2(\mathcal{W})$.

Definition 3.3. A maximal nonpositive S-reducing subspace $\mathfrak{W}^\dagger$ of $k^2(\mathcal{W})$ is anti-causal if it is true for some fundamental decomposition $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ of $\mathcal{W}$ that

$$w^\dagger(\cdot) \in \mathfrak{W}^\dagger \quad \text{and} \quad P_{\ell^2_+(\mathcal{Y})} w = 0 \quad \Rightarrow \quad \pi_+ w(\cdot) = 0. \tag{3.5}$$

Note, in particular, that the projection here is onto the *negative* component in the fundamental decomposition $k^2_+(\mathcal{W}) = -\ell^2_+(\mathcal{Y})\ [\dot{+}]\ \ell^2_+(\mathcal{U})$, and that π_- in Definition 2.6 now has been replaced by π_+.

Theorem 3.4. *Let $\mathfrak{W}^{\Sigma}_{\mathrm{past}}$, $\mathfrak{W}^{\Sigma}_{\mathrm{full}}$, and $\mathfrak{W}^{\Sigma}_{\mathrm{fut}}$ be the past, full, and future behaviors of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$, and let $\mathfrak{W}^{\Sigma^\dagger}_{\mathrm{past}}$, $\mathfrak{W}^{\Sigma^\dagger}_{\mathrm{past}}$, $\mathfrak{W}^{\Sigma^\dagger}_{\mathrm{past}}$ be the past, full, and future behaviors of the anti-passive dual $\Sigma^\dagger = (V^{[\perp]}; \mathcal{X}, \mathcal{W})$. Then*

(1) $\mathfrak{W}^{\Sigma^\dagger}_{\mathrm{past}}$ *is a maximal nonpositive S^*_--invariant subspace of $k^2_-(\mathcal{W})$.*
(2) $\mathfrak{W}^{\Sigma^\dagger}_{\mathrm{full}}$ *is a maximal nonpositive anti-causal S-reducing subspace of $k^2(\mathcal{W})$.*
(3) $\mathfrak{W}^{\Sigma^\dagger}_{\mathrm{fut}}$ *is a maximal nonpositive S^*_+-invariant subspace of $k^2_+(\mathcal{W})$.*
(4) $\mathfrak{W}^{\Sigma^\dagger}_{\mathrm{past}} = \mathfrak{W}^{\Sigma^\dagger}_{\mathrm{full}} \cap k^2_-(\mathcal{W})$.
(5) $\mathfrak{W}^{\Sigma^\dagger}_{\mathrm{full}} = \bigvee_{n\in\mathbb{Z}^+} S^n \mathfrak{W}^{\Sigma^\dagger}_{\mathrm{past}}$.
(6) $\mathfrak{W}^{\Sigma^\dagger}_{\mathrm{fut}} = \pi_+ \mathfrak{W}^{\Sigma^\dagger}_{\mathrm{full}}$.
(7) $\mathfrak{W}^{\Sigma^\dagger}_{\mathrm{full}} = \bigcap_{n\in\mathbb{Z}^+} \{w(\cdot) \in k^2(\mathcal{W}) \mid \pi_+ S^n w \in \mathfrak{W}^{\Sigma^\dagger}_{\mathrm{fut}}\}$.
(8) $\mathfrak{W}^{\Sigma^\dagger}_{\mathrm{past}} = (\mathfrak{W}^{\Sigma}_{\mathrm{past}})^{[\perp]}$, $\mathfrak{W}^{\Sigma^\dagger}_{\mathrm{full}} = (\mathfrak{W}^{\Sigma}_{\mathrm{full}})^{[\perp]}$, *and* $\mathfrak{W}^{\Sigma^\dagger}_{\mathrm{fut}} = (\mathfrak{W}^{\Sigma}_{\mathrm{fut}})^{[\perp]}$.

Proof. Claims (1)–(7) are proved in the same way as in Theorem 2.8, either by repeating essentially the same argument with Σ replaced by $\Sigma^\dagger$, or by applying Theorem 2.8 to the passive dual Σ_* of Σ and then doing a time reflection and replacing $-\mathcal{W}$ by $\mathcal{W}$ to get the anti-passive dual $\Sigma^\dagger$. If one chooses the second alternative one needs to know the connections between $\mathfrak{W}^{[\perp]}$, $\mathfrak{W}^{[\perp]}_+$, and $\mathfrak{W}^{[\perp]}_-$ explained in Lemma 3.5 below.

The three identities in claim (8) are in principle proved in the same way, so we only prove one of these. If $(x(\cdot), w(\cdot))$ and let $(x^\dagger(\cdot), w^\dagger(\cdot))$ are stable externally generated trajectories of Σ and $\Sigma^\dagger$, respectively, then by Lemma 3.2, $[w(\cdot), w^\dagger(\cdot)]_{k^2(\mathcal{W})} = 0$. This implies that $\mathfrak{W}^{\Sigma^\dagger}_{\mathrm{full}} \subset (\mathfrak{W}^{\Sigma}_{\mathrm{full}})^{[\perp]}$. Since $\mathfrak{W}^{\Sigma^\dagger}_{\mathrm{full}}$ is maximal nonpositive and $(\mathfrak{W}^{\Sigma}_{\mathrm{full}})^{[\perp]}$ is nonpositive, this implies that $\mathfrak{W}^{\Sigma^\dagger}_{\mathrm{full}} = (\mathfrak{W}^{\Sigma}_{\mathrm{full}})^{[\perp]}$. $\square$

Lemma 3.5. *Let $\mathfrak{W}$ be a closed subspace of $k^2(\mathcal{W})$, and define $\mathfrak{W}_\pm$ by* (2.15). *Then*

$$\mathfrak{W}_-^{[\perp]} = \mathfrak{W}^{[\perp]} \cap k_-^2(\mathcal{W}), \qquad \mathfrak{W}_+^{[\perp]} = \overline{\pi_+ \mathfrak{W}^{[\perp]}}. \tag{3.6}$$

Conversely, if (3.6) *hold, then $\mathfrak{W}_+ = \mathfrak{W} \cap k_+^2(\mathcal{W})$ and $\overline{\mathfrak{W}_-} = \overline{\pi_- \mathfrak{W}}$. Here $\mathfrak{W}_\pm^{[\perp]}$ is the orthogonal companion of $\mathfrak{W}_\pm$ in $k_\pm^2(\mathcal{W})$ and $\mathfrak{W}^{[\perp]}$ is the orthogonal companion of $\mathfrak{W}$ in $k^2(\mathcal{W})$.*

Proof. For each $w_- \in k_-^2(\mathcal{W})$ and $w \in k^2(\mathcal{W})$ we have $[w_-, w]_{k^2(\mathcal{W})} = [w_-, \pi_- w]_{k_-^2(\mathcal{W})}$. This gives

$$\begin{aligned}
(\pi_- \mathfrak{W})_-^{[\perp]} &= \{ w_- \in k_-^2(\mathcal{W}) \mid [w_-, w_p]_{k_-^2(\mathcal{W})} = 0 \text{ for all } w_p \in \pi_- \mathfrak{W} \} \\
&= \{ w_- \in k_-^2(\mathcal{W}) \mid [w_-, \pi_- w]_{k^2(\mathcal{W})} = 0 \text{ for all } w \in \mathfrak{W} \} \\
&= \{ w_- \in k_-^2(\mathcal{W}) \mid [w_-, w]_{k^2(\mathcal{W})} = 0 \text{ for all } w \in \mathfrak{W} \} \\
&= \mathfrak{W}^{[\perp]} \cap k_-^2(\mathcal{W}).
\end{aligned}$$

Thus, if $\mathfrak{W}_- = \pi_- \mathfrak{W}$, then $\mathfrak{W}_-^{[\perp]} = \mathfrak{W}^{[\perp]} \cap k_-^2(\mathcal{W})$. Conversely, if $\mathfrak{W}_-^{[\perp]} = \mathfrak{W}^{[\perp]} \cap k_-^2(\mathcal{W})$, then by the above computation, $\mathfrak{W}_-^{[\perp]} = (\pi_- \mathfrak{W})^{[\perp]}$, and hence

$$\overline{\mathfrak{W}_-} = \big(\mathfrak{W}_-^{[\perp]}\big)^{[\perp]} = \big((\pi_- \mathfrak{W})^{[\perp]}\big)^{[\perp]} = \overline{\pi_- \mathfrak{W}^{[\perp]}}.$$

For the second half of (3.6) we use essentially the same computation to get (recall that $(\mathfrak{W}^{[\perp]})^{[\perp]} = \mathfrak{W}$ since $\mathfrak{W}$ is closed)

$$\begin{aligned}
\big(\pi_+ \mathfrak{W}^{[\perp]}\big)^{[\perp]} &= \{ w_+ \in k_+^2(\mathcal{W}) \mid [w_+, w_f]_{k_-^2(\mathcal{W})} = 0 \text{ for all } w_f \in \pi_+ \mathfrak{W}^{[\perp]} \} \\
&= \{ w_+ \in k_+^2(\mathcal{W}) \mid [w_+, \pi_+ w]_{k^2(\mathcal{W})} = 0 \text{ for all } w \in \mathfrak{W}^{[\perp]} \} \\
&= \{ w_+ \in k_+^2(\mathcal{W}) \mid [w_+, w]_{k^2(\mathcal{W})} = 0 \text{ for all } w \in \mathfrak{W}^{[\perp]} \} \\
&= \big(\mathfrak{W}^{[\perp]}\big)^{[\perp]} \cap k_+^2(\mathcal{W}) = \mathfrak{W} \cap k_+^2(\mathcal{W}).
\end{aligned}$$

Thus, if $\mathfrak{W}_+ = \mathfrak{W} \cap k_+^2(\mathcal{W})$, then $\mathfrak{W}_+^{[\perp]} = ((\pi_+ \mathfrak{W}^{[\perp]})^\perp)^{[\perp]} = \overline{\pi_+ \mathfrak{W}^{[\perp]}}$. Conversely, if $\mathfrak{W}_+^{[\perp]} = \overline{\pi_+ \mathfrak{W}^{[\perp]}}$, then the above computation together with the fact that $\mathfrak{W}_+$ is closed gives

$$\mathfrak{W}_+ = \big((\mathfrak{W}_+)^{[\perp]}\big)^{[\perp]} = \big(\overline{\pi_+ \mathfrak{W}^{[\perp]}}\big)^{[\perp]} = \big(\pi_+ \mathfrak{W}^{[\perp]}\big)^{[\perp]} = \mathfrak{W} \cap k_+^2(\mathcal{W}). \qquad \square$$

Definition 3.6. Let $\mathcal{W}$ be a Kreĭn space.

(1) A maximal nonpositive S_-^*-invariant subspace of $k_-^2(\mathcal{W})$ is called a *anti-passive past behavior* on the Kreĭn (signal) space $\mathcal{W}$.
(2) A maximal nonpositive S-reducing anti-causal subspace of $k^2(\mathcal{W})$ is called a *anti-passive full behavior* on the (signal) space $\mathcal{W}$.
(3) A maximal nonpositive S_+^*-invariant subspace of $k_+^2(\mathcal{W})$ is called a *anti-passive future behavior* on the (signal) space $\mathcal{W}$.

Theorem 3.7. *Let $\mathcal{W}$ be a Kreĭn space.*

(1) *If $\mathfrak{W}^\dagger$ is an anti-passive full behavior on $\mathfrak{W}$, and if we define $\mathfrak{W}^\dagger_+$ and $\mathfrak{W}^\dagger_-$ by*

$$\mathfrak{W}^\dagger_- := \mathfrak{W}^\dagger \cap k^2_-(\mathcal{W}), \qquad \mathfrak{W}^\dagger_+ := \pi_+\mathfrak{W}^\dagger, \tag{3.7}$$

then $\mathfrak{W}^\dagger_-$ and $\mathfrak{W}^\dagger_+$ are anti-passive past and future behaviors on $\mathcal{W}$, respectively, and $\mathfrak{W}^\dagger$ can be recovered from $\mathfrak{W}^\dagger_-$ and from $\mathfrak{W}^\dagger_+$ by the formulas

$$\mathfrak{W}^\dagger = \bigvee_{n\in\mathbb{Z}^+} S^n\mathfrak{W}^\dagger_-, \tag{3.8}$$

$$\mathfrak{W}^\dagger = \bigcap_{n\in\mathbb{Z}^+} \bigl\{w(\cdot)\in k^2(\mathcal{W}) \bigm| \pi_-S^n w \in \mathfrak{W}^\dagger_+\bigr\}. \tag{3.9}$$

(2) *If $\mathfrak{W}^\dagger_-$ is an anti-passive past behavior on $\mathcal{W}$, and if we define $\mathfrak{W}^\dagger$ by* (3.8)*, then $\mathfrak{W}^\dagger$ is an anti-passive full behavior on $\mathcal{W}^\dagger$ and $\mathfrak{W}^\dagger_+ = \mathcal{W}^\dagger \cap k^2_-(\mathcal{W})$.*

(3) *If $\mathfrak{W}^\dagger_+$ is an anti-passive future behavior on $\mathcal{W}$, and if we define $\mathfrak{W}^\dagger$ by* (3.9)*, then $\mathfrak{W}^\dagger$ is an anti-passive full behavior on $\mathcal{W}$ and $\mathfrak{W}^\dagger_+ = \pi_+\mathfrak{W}^\dagger$.*

Proof. This is the anti-passive version of Theorem 2.11. □

Lemma 3.8. *Let $\mathfrak{W}_+$ be a passive future behavior on a Kreĭn space $\mathcal{W}$. Then the set of all $w^\dagger(\cdot)\in\mathfrak{W}^{[\perp]}_+$ with finite support* (*i.e., $w^\dagger(k)=0$ for all k in some interval* $[m,\infty)$) *is a dense subspace of $\mathfrak{W}^{[\perp]}_+$.*

Proof. The set $\mathfrak{W}^\dagger_+ := \mathfrak{W}^{[\perp]}_+$ is an anti-passive future behavior on $\mathcal{W}$. By (3.7) and (3.8),

$$\mathfrak{W}^\dagger_+ = \pi_+\mathfrak{W}^\dagger = \pi_+ \bigvee_{n\in\mathbb{Z}^+} S^n\bigl(\mathfrak{W}\cap k^2_-(\mathcal{W})\bigr) = \bigvee_{n\in\mathbb{Z}^+} \pi_+S^n\bigl(\mathfrak{W}\cap k^2_-(\mathcal{W})\bigr),$$

where each sequence in $\pi_+S^{-n}(\mathfrak{W}^\dagger\cap k^2_-(\mathcal{W}))$ has finite support. □

In some cases the following simple lemma is also useful.

Lemma 3.9. *Let $\mathfrak{W}$ be a closed S-reducing subspace of $k^2(\mathcal{W})$, and define $\mathfrak{W}_\pm$ by* (2.15)*. Then*

$$S_+\mathfrak{W}_+ \subset \mathfrak{W}_+, \qquad S^*_+\mathfrak{W}^{[\perp]}_+ = \mathfrak{W}^{[\perp]}_+, \tag{3.10}$$

$$S_-\mathfrak{W}_- = \mathfrak{W}_-, \qquad S^*_-\mathfrak{W}^{[\perp]}_- \subset \mathfrak{W}^{[\perp]}_-. \tag{3.11}$$

Proof. The two inclusions in (3.10) and (3.11) are obvious. That the equality in (3.11) holds follows from (2.21). To prove the equality in (3.10) we use Lemma 3.5 and the fact that $\mathfrak{W}^{[\perp]}$ is S-reducing to compute

$$S^*_+\mathfrak{W}^{[\perp]}_+ = \pi_+S^{-1}\overline{\pi_+\mathfrak{W}^{[\perp]}} = \pi_+\overline{\pi_{[-1,\infty)}S^{-1}\mathfrak{W}^{[\perp]}} = \overline{\pi_+\mathfrak{W}^{[\perp]}} = \mathfrak{W}^{[\perp]}_+. \qquad \square$$

Remark 3.10. The following analogue of Remark 2.15 is true: *If $\mathfrak{W}$ is a S-reducing subspace of $k^2(\mathcal{W})$ with the property that $\mathfrak{W}^{[\perp]}$ is nonpositive and that $\pi_-\mathfrak{W}$ contains some maximal nonnegative S_--invariant subspace $\mathfrak{W}_-$ of $k^2_-(\mathcal{W})$, then $\mathfrak{W}$ is given by* (2.17). This can be proved by applying the extended version of claim (2) to the orthogonal companion $\mathfrak{W}^{[\perp]}$ of $\mathfrak{W}$, using Lemma 3.5.

4. The Hilbert spaces $\mathcal{H}(\mathfrak{W}_+)$ and $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$

In this section we shall present two special Hilbert spaces that play a central role throughout the rest of this article. Among others, they will be used as the state spaces of two of our canonical realizations of a passive behavior. These two spaces are special cases of the Hilbert space $\mathcal{H}(\mathcal{Z})$ constructed in [6], where $\mathcal{Z}$ is a maximal nonnegative subspace of a Kreĭn space $\mathcal{K}$. We begin with a short review of those results in [6] which are relevant here.

The Hilbert space $\mathcal{H}(\mathcal{Z})$

Let $\mathcal{Z}$ be a maximal nonnegative subspace of the Kreĭn space $\mathcal{K}$, and let $\mathcal{K}/\mathcal{Z}$ be the quotient of $\mathcal{K}$ modulo $\mathcal{Z}$. We define $\mathcal{H}(\mathcal{Z})$ by

$$\mathcal{H}(\mathcal{Z}) = \{h \in \mathcal{K}/\mathcal{Z} \mid \sup\{-[x,x]_{\mathcal{K}} \mid x \in h\} < \infty\}. \tag{4.1}$$

It turns out that $\sup\{-[x,x]_{\mathcal{K}} \mid x \in h\} \geqslant 0$ for all $h \in \mathcal{H}(\mathcal{Z})$, that $\mathcal{H}(\mathcal{Z})$ is a subspace of $\mathcal{K}$, that $\mathcal{H}(\mathcal{Z})$ is a Hilbert space with the norm

$$\|h\|_{\mathcal{H}(\mathcal{Z})} = \left(\sup\{-[x,x]_{\mathcal{K}} \mid x \in h\}\right)^{1/2}, \quad h \in \mathcal{H}(\mathcal{Z}), \tag{4.2}$$

and that $\mathcal{H}(\mathcal{Z})$ is continuously contained in $\mathcal{X}/\mathcal{Z}$. We denote the equivalence class $h \in \mathcal{K}/\mathcal{Z}$ that contains a particular vector $x \in \mathcal{K}$ by $h = x + \mathcal{Z}$. Thus, with this notation, (4.1) and (4.2) can be rewritten in the form

$$\mathcal{H}(\mathcal{Z}) = \{x + \mathcal{Z} \in \mathcal{K}/\mathcal{Z} \mid \|x + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} < \infty\}, \tag{4.3}$$

$$\|x + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} = \left(\sup\{-[x+z, x+z]_{\mathcal{K}} \mid z \in \mathcal{Z}\}\right), \quad x \in \mathcal{H}(\mathcal{Z}). \tag{4.4}$$

A very important (and easily proved) fact is that if we define

$$\mathcal{H}^0(\mathcal{Z}) := \{z^\dagger + \mathcal{Z} \mid z^\dagger \in \mathcal{Z}^{[\perp]}\}, \tag{4.5}$$

then $\mathcal{H}^0(\mathcal{Z})$ is a subspace of $\mathcal{H}(\mathcal{Z})$. However, even more is true: $\mathcal{H}^0(\mathcal{Z})$ is a *dense subspace* of $\mathcal{H}(\mathcal{Z})$, and for every $z^\dagger \in \mathcal{Z}^{[\perp]}$ it is true that

$$\|z^\dagger + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} = -[z^\dagger, z^\dagger]_{\mathcal{K}}, \quad z^\dagger \in \mathcal{Z}^{[\perp]}. \tag{4.6}$$

Furthermore, it is easy to compute the inner product in $\mathcal{H}(\mathcal{Z})$ of a vector in $\mathcal{H}^0(\mathcal{Z})$ with any vector in $\mathcal{H}(\mathcal{Z})$. To explain how this is done we introduce the notation

$$\mathcal{K}(\mathcal{Z}) = \{x \in \mathcal{K} \mid x + \mathcal{Z} \in \mathcal{H}(\mathcal{Z})\}. \tag{4.7}$$

Thus, $\mathcal{H}(\mathcal{Z}) = \{x + \mathcal{Z} \mid x \in \mathcal{K}(\mathcal{Z})\}$, and $\mathcal{K}(\mathcal{Z})$ is the domain of the restriction of the quotient map $\pi_{\mathcal{Z}} := x \mapsto x + \mathcal{Z}$ to those $x \in \mathcal{X}$ for which $\pi_{\mathcal{Z}} x \in \mathcal{H}(\mathcal{Z})$. Let us denote this restriction by R and interpret it as a map $\mathcal{K} \to \mathcal{H}(\mathcal{Z})$ with domain $\mathcal{K}(\mathcal{Z})$. Then R is a closed and surjective linear operator; this follows from the definition of $\mathcal{K}(\mathcal{Z})$ and the fact that $\mathcal{H}(\mathcal{Z})$ is continuously contained in $\mathcal{X}/\mathcal{Z}$ (for the closedness it is important that we use the $\mathcal{H}(\mathcal{Z})$-norm in the range space). In particular, R has a bounded right-inverse $\mathcal{H}(\mathcal{Z}) \to \mathcal{K}$. Moreover, if $x_n \in \mathcal{K}(\mathcal{Z})$ and $x_n + \mathcal{Z} \to x + \mathcal{Z}$ for some $x \in \mathcal{K}(\mathcal{Z})$, then there exists a sequence $z_n \in \mathcal{Z}$ such that $x_n + z_n \to x$ in $\mathcal{K}$; this is true because $\mathcal{H}(\mathcal{Z})$ is continuously contained in $\mathcal{X}/\mathcal{Z}$ and $\pi_{\mathcal{Z}}$ has a bounded right-inverse. The rule for computing the inner product of a vector $z^\dagger + \mathcal{Z} \in \mathcal{H}^0(\mathcal{Z})$ and a vector $x + \mathcal{Z} \in \mathcal{H}(\mathcal{Z})$ is the following:

$$\left(z^\dagger + \mathcal{Z}, x + \mathcal{Z}\right)_{\mathcal{H}(\mathcal{Z})} = -\left[z^\dagger, x\right]_{\mathcal{K}}, \quad z^\dagger \in \mathcal{Z}^{[\perp]},\ x \in \mathcal{K}(\mathcal{Z}). \tag{4.8}$$

See [6] for more details.

In this article we shall need the results cited above with either $\mathcal{K} = k_+^2(\mathcal{W})$ for some Kreĭn space $\mathcal{W}$ and $\mathcal{Z} = \mathfrak{W}_+$ for some passive future behavior $\mathfrak{W}_+$ on $\mathcal{W}$, or $\mathcal{K} = -k_-^2(\mathcal{W})$ and $\mathcal{Z} = \mathfrak{W}_-^{[\perp]}$ for some passive past behavior $\mathfrak{W}_-$ on $\mathcal{W}$, interpreted as a maximal nonnegative subspace of $-k_-^2(\mathcal{W})$.

The Hilbert space $\mathcal{H}(\mathfrak{W}_+)$

Let $\mathfrak{W}_+$ be a given passive future behavior on a Kreĭn signal space $\mathcal{W}$, i.e., $\mathfrak{W}_+$ is a maximal nonnegative S_+-invariant subspace of $k_+^2(\mathcal{W})$. We take $\mathcal{K} = k_+^2(\mathcal{W})$ and $\mathcal{Z} = \mathfrak{W}_+$ in the discussion above, and adapting our earlier formulas to this case we get the following result.

Theorem 4.1. *Let $\mathfrak{W}_+$ be a passive future behavior on the Kreĭn space $k_+^2(\mathcal{W})$. Define*

$$\mathcal{H}(\mathfrak{W}_+) = \left\{h_+ \in k_+^2(\mathcal{W})/\mathfrak{W}_+ \;\middle|\; \sup\left\{-[w_+, w_+]_{k_+^2(\mathcal{W})} \;\middle|\; w_+ \in h_+\right\} < \infty\right\}, \tag{4.9}$$

and define $\|\cdot\|_{\mathcal{H}(\mathfrak{W}_+)}$ by

$$\|h_+\|_{\mathcal{H}(\mathfrak{W}_+)} = \left(\sup\left\{-[w_+, w_+]_{k_+^2(\mathcal{W})} \;\middle|\; w_+ \in h_+\right\}\right)^{1/2}, \quad h_+ \in \mathcal{H}(\mathfrak{W}_+). \tag{4.10}$$

Then $\mathcal{H}(\mathfrak{W}_+)$ is a Hilbert space with the norm $\|\cdot\|_{\mathcal{H}(\mathfrak{W}_+)}$ that is continuously contained in $k_+^2(\mathcal{W})/\mathfrak{W}_+$. The set

$$\mathcal{H}^0(\mathfrak{W}_+) := \left\{w_+^\dagger + \mathfrak{W}_+ \;\middle|\; w_+^\dagger \in \mathfrak{W}_+^{[\perp]}\right\} \tag{4.11}$$

is a dense subspace of $\mathcal{H}(\mathfrak{W}_+)$, and

$$\left\|w_+^\dagger + \mathfrak{W}_+\right\|_{\mathcal{H}(\mathfrak{W}_+)}^2 = -\left[w_+^\dagger(\cdot), w_+^\dagger(\cdot)\right]_{k_+^2(\mathcal{W})}, \quad w_+^\dagger \in \mathfrak{W}_+^{[\perp]}. \tag{4.12}$$

The set

$$\mathcal{K}(\mathfrak{W}_+) = \left\{w_+(\cdot) \in k_+^2(\mathcal{W}) \;\middle|\; w_+(\cdot) + \mathfrak{W}_+ \in \mathcal{H}(\mathfrak{W}_+)\right\} \tag{4.13}$$

is a subspace of $k_+^2(\mathcal{W})$, *and*

$$\begin{aligned}&\big(w_+^\dagger(\cdot)+\mathfrak{W}_+, w(\cdot)+\mathfrak{W}_+\big)_{\mathcal{H}(\mathfrak{W}_+)} = -\big[w_+^\dagger(\cdot), w_+(\cdot)\big]_{k_+^2(\mathcal{W})},\\ &\quad \textit{if } w_+^\dagger(\cdot)\in\mathfrak{W}_+^{[\perp]} \textit{ and } w_+(\cdot)\in\mathcal{K}(\mathfrak{W}_+).\end{aligned}\tag{4.14}$$

The restriction R_+ *of the quotient map* $\pi_{\mathfrak{W}_+}: w_+(\cdot)\mapsto w_+(\cdot)+\mathfrak{W}_+$ *to those* $w_+(\cdot)\in k_+^2(\mathcal{W})$ *for which* $\pi_{\mathfrak{W}_+}w_+\in\mathcal{H}(\mathfrak{W}_+)$, *regarded as an operator* $k_+^2(\mathcal{W})\to\mathcal{H}(\mathfrak{W}_+)$, *is closed and surjective with domain* $\mathcal{K}(\mathfrak{W}_+)$, *and it has a bounded right-inverse. Moreover, if* $w_+^k(\cdot)\in\mathcal{K}(\mathfrak{W}_+)$ *and* $w_+^k(\cdot)+\mathfrak{W}_+\to w_+(\cdot)+\mathfrak{W}_+$ *in* $\mathcal{H}(\mathfrak{W}_+)$ *for some* $w_+(\cdot)\in\mathcal{K}(\mathfrak{W}_+)$, *then there exists a sequence* $z_+^k(\cdot)\in\mathfrak{W}_+$ *such that* $w_+^k(\cdot)+z_+^k(\cdot)\to w_+(\cdot)$ *in* $k_+^2(\mathcal{W})$.

Lemma 4.2. *Let* $\mathfrak{W}_+$ *be a passive future behavior on the Kreĭn space* $\mathcal{W}$. *Then the set*

$$\mathcal{H}_0^0(\mathfrak{W}_+) := \big\{w_+^\dagger+\mathfrak{W}_+ \ \big|\ w_+^\dagger\in\mathfrak{W}_+^{[\perp]} \textit{ and } w_+^\dagger \textit{ has finite support}\big\}$$

(*which is contained in* $\mathcal{H}^0(\mathfrak{W}_+)$) *is a dense subspace of* $\mathcal{H}(\mathfrak{W}_+)$.

Proof. Let $w_+^\dagger\in\mathfrak{W}_+^{[\perp]}$. Then by Lemma 3.8, there exists a sequence $w_+^k(\cdot)\in\mathfrak{W}_+^{[\perp]}$, where each w_+^k has finite support, such that $w_+^k\to w_+^\dagger$ in $k_+^2(\mathcal{W})$ as $k\to\infty$. This implies that $[w_+^k-w_+^\dagger, w_+^k-w_+^\dagger]_{k_+^2(\mathcal{W})}\to 0$ as $n\to\infty$, and according to (4.12), this means that $w_+^k+\mathfrak{W}_+\to w_+^\dagger+\mathfrak{W}_+$ in $\mathcal{H}(\mathfrak{W}_+)$ as $n\to\infty$. Since $\mathcal{H}^0(\mathfrak{W}_+)$ is dense in $\mathcal{H}(\mathfrak{W}_+)$, this proves the lemma. $\square$

Lemma 4.3. *If* $w_+(\cdot)\in\mathcal{K}(\mathfrak{W}_+)$, *where* $\mathfrak{W}_+$ *is a passive future behavior on the Kreĭn space* $\mathcal{W}$, *then* $S_+^*w_+\in\mathcal{K}(\mathfrak{W}_+)$ *and*

$$\big\|S_+^*w_++\mathfrak{W}_+\big\|_{\mathcal{H}(\mathfrak{W}_+)}^2 \leqslant \|w_++\mathfrak{W}_+\|_{\mathcal{H}(\mathfrak{W}_+)}^2 + \big[w_+(0), w_+(0)\big]_{\mathcal{W}}.\tag{4.15}$$

If $w_+(\cdot)\in\mathfrak{W}_+^{[\perp]}$, *then* $w_+(\cdot)\in\mathcal{K}(\mathfrak{W}_+)$ *and* (4.15) *holds as an equality.*

Proof. We have for all $w_+(\cdot)\in\mathcal{K}(\mathfrak{W}_+)$ and all $z\in\mathfrak{W}_+$,

$$\begin{aligned}-\big[S_+^*w_++z, S_+^*w_++z\big]_{k_+^2(\mathcal{W})} &= -\big[S_+^*(w_++S_+z), S_+^*(w_++S_+z)\big]_{k_+^2(\mathcal{W})}\\ &= -[w_++S_+z, w_++S_+z]_{k_+^2(\mathcal{W})} + \big[w_+(0), w_+(0)\big]_{\mathcal{W}}\\ &\leqslant \|w_++\mathfrak{W}_+\|_{\mathcal{H}(\mathfrak{W}_+)}^2 + \big[w_+(0), w_+(0)\big]_{\mathcal{W}}.\end{aligned}$$

From here we get (4.15) by taking the supremum over all $z\in\mathfrak{W}_+$. If $w_+\in\mathfrak{W}_+^{[\perp]}$, then $w_++\mathfrak{W}_+\in\mathcal{H}^0(\mathfrak{W}_+)\subset\mathcal{H}(\mathfrak{W}_+)$, and by (4.10),

$$\begin{aligned}&\big\|S_+^*w_++\mathfrak{W}_+\big\|_{\mathcal{H}(\mathfrak{W}_+)}^2 - \|w_++\mathfrak{W}_+\|_{\mathcal{H}(\mathfrak{W}_+)}^2\\ &\quad = -\big[S_+^*w_+, S_+^*w_+\big]_{k_+^2(\mathcal{W})} + [w_+, w_+]_{k_+^2(\mathcal{W})} = \big[w_+(0), w_+(0)\big]_{\mathcal{W}}. \qquad \square\end{aligned}$$

The Hilbert space $\mathcal{H}(\mathfrak{W}_-^{\perp})$

Let $\mathfrak{W}_-$ be a given passive past behavior on a Kreĭn signal space $\mathcal{W}$, i.e., $\mathfrak{W}_-$ is a maximal nonnegative S_--invariant subspace of $k_-^2(\mathcal{W})$. Then $\mathfrak{W}_-^{[\perp]}$ is a maximal nonpositive S_-^*-invariant subspace of $k_-^2(\mathcal{W})$, and hence it can be interpreted as a maximal nonnegative S_-^*-invariant subspace of the anti-space $-k_-^2(\mathcal{W})$. This time we take $\mathcal{K} = -k_-^2(\mathcal{W})$ and $\mathcal{Z} = \mathfrak{W}_-^{[\perp]}$ in the definition of $\mathcal{H}(\mathcal{Z})$. Adapting our earlier formulas to this case we get the following result.

Theorem 4.4. *Let $\mathfrak{W}_-$ be a passive past behavior on the Kreĭn space $k_-^2(\mathcal{W})$, and interpret $\mathfrak{W}_-^{[\perp]}$ as a maximal nonnegative S_-^*-invariant subspace of the anti-space $-k_-^2(\mathcal{W})$. Define*

$$\mathcal{H}\big(\mathfrak{W}_-^{[\perp]}\big) = \big\{h_- \in -k_-^2(\mathcal{W})/\mathfrak{W}_-^{[\perp]} \bigm| \sup\big\{\big[w_-(\cdot), w_-(\cdot)\big]_{k_-^2(\mathcal{W})} \bigm| w_-(\cdot) \in h_-\big\} < \infty\big\}, \tag{4.16}$$

and define $\|\cdot\|_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})}$ by

$$\|h_-\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} = \sup\big\{\big[w_-(\cdot), w_-(\cdot)\big]_{k_-^2(\mathcal{W})} \bigm| w_-(\cdot) \in h_-\big\}. \tag{4.17}$$

Then $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ is a Hilbert space with the norm $\|\cdot\|_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})}$ that is continuously contained in $-k_-^2(\mathcal{W})/\mathfrak{W}_-^{[\perp]}$. The set

$$\mathcal{H}^0\big(\mathfrak{W}_-^{[\perp]}\big) = \big\{w_-(\cdot) + \mathfrak{W}_-^{[\perp]} \bigm| w_-(\cdot) \in \mathfrak{W}_-\big\} \tag{4.18}$$

is a dense subspace of $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$, and

$$\big\|w_- + \mathfrak{W}_-^{[\perp]}\big\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} = \big[w_-(\cdot), w_-(\cdot)\big]_{k_-^2(\mathcal{W})}, \quad w_-(\cdot) \in \mathfrak{W}_-. \tag{4.19}$$

The set

$$\mathcal{K}\big(\mathfrak{W}_-^{[\perp]}\big) = \big\{w_-(\cdot) \in k_-^2(\mathcal{W}) \bigm| w_-(\cdot) + \mathfrak{W}_-^{[\perp]} \in \mathcal{H}\big(\mathfrak{W}_-^{[\perp]}\big)\big\} \tag{4.20}$$

is a subspace of $k_-^2(\mathcal{W})$, and

$$\begin{aligned}&\big(w_-(\cdot) + \mathfrak{W}_-^{[\perp]}, v_-(\cdot) + \mathfrak{W}_-^{[\perp]}\big)_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} = \big[w_-(\cdot), v_-(\cdot)\big]_{k_-^2(\mathcal{W})},\\ &\quad \text{if } w_-(\cdot) \in \mathfrak{W}_- \text{ and } v_-(\cdot) \in \mathcal{K}\big(\mathfrak{W}_-^{[\perp]}\big).\end{aligned} \tag{4.21}$$

The restriction R_- of the quotient map $\pi_{\mathfrak{W}_-^{[\perp]}} : w_-(\cdot) \mapsto w_-(\cdot) + \mathfrak{W}_-^{[\perp]}$ to those $w_-(\cdot) \in k_-^2(\mathcal{W})$ for which $\pi_{\mathfrak{W}_-^{[\perp]}} w_- \in \mathcal{H}(\mathfrak{W}_-^{[\perp]})$, regarded as an operator $k_-^2(\mathcal{W}) \to \mathcal{H}(\mathfrak{W}_-^{[\perp]})$, is closed and surjective with domain $\mathcal{K}(\mathfrak{W}_-^{[\perp]})$, and it has a bounded right-inverse. Moreover, if $w_-^k(\cdot) \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$ and $w_-^k(\cdot) + \mathfrak{W}_-^{[\perp]} \to w_-(\cdot) + \mathfrak{W}_-^{[\perp]}$ in $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ for some $w_-(\cdot) \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$, then there exists a sequence $z_-^k(\cdot) \in \mathfrak{W}_-^{[\perp]}$ such that $w_-^k(\cdot) + z_-^k(\cdot) \to w_-(\cdot)$ in $k_-^2(\mathcal{W})$.

Lemma 4.5. *Let $\mathfrak{W}_-$ be a passive past behavior on the Kreĭn space $\mathcal{W}$. Then the set*

$$\mathcal{H}_0^0 := \{ w_- + \mathfrak{W}_-^{[\perp]} \mid w_- \in \mathfrak{W}_- \text{ and } w_- \text{ has finite support} \}$$

(*which is contained in $\mathcal{H}^0(\mathfrak{W}_-^{[\perp]})$*) *is a dense subspace of $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$.*

Proof. The proof of this lemma is analogous to the proof of Lemma 4.2. □

Lemma 4.6. *If $w_-(\cdot) \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$, then $S_- w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$ and*

$$\| S_- w_- + \mathfrak{W}_-^{[\perp]} \|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} \leqslant \| w_- + \mathfrak{W}_-^{[\perp]} \|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} - [w_-(-1), w_-(-1)]_{\mathcal{W}}. \tag{4.22}$$

If $w_-(\cdot) \in \mathfrak{W}_-$, then $w_-(\cdot) \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$ and (4.22) *holds as an equality.*

Proof. The proof of this lemma is analogous to the proof of Lemma 4.3. □

5. The output and input maps

The output map $\mathfrak{C}_\Sigma$

We begin by presenting the output map of a passive s/s system.

Lemma 5.1. *Let $\Sigma = (V; \mathcal{X}; \mathcal{W})$ be a passive s/s system with future behavior $\mathfrak{W}_{\mathrm{fut}}$. If $(x(\cdot), w(\cdot))$ is a stable future trajectory of Σ, then*

$$w(\cdot) \in \mathcal{K}(\mathfrak{W}_{\mathrm{fut}}) \quad \text{and} \quad \| w(\cdot) + \mathfrak{W}_{\mathrm{fut}} \|_{\mathcal{H}(\mathfrak{W}_{\mathrm{fut}})} \leqslant \| x(0) \|_{\mathcal{X}}. \tag{5.1}$$

Proof. Let $(x(\cdot), w(\cdot))$ be a stable future trajectory of Σ, let $z(\cdot) \in \mathfrak{W}_{\mathrm{fut}}$, and let $(x_1(\cdot), z(\cdot))$ be the corresponding externally generated stable future trajectory of Σ. Then $(x(\cdot) + x_1(\cdot), w(\cdot) + z(\cdot))$ is a stable future trajectory of Σ, and by (1.8),

$$-[w(\cdot) + z(\cdot), w(\cdot) + z(\cdot)]_{k_+^2(\mathcal{W})} \leqslant \| x(0) + x_1(0) \|^2_{\mathcal{X}} = \| x(0) \|^2_{\mathcal{X}}.$$

Taking the supremum over all $z \in \mathfrak{W}_{\mathrm{fut}}$ we find that (5.1) holds. □

Lemma 5.2. *Let $\Sigma = (V; \mathcal{X}; \mathcal{W})$ be a passive s/s system with future behavior $\mathfrak{W}_{\mathrm{fut}}$. Then the formula*

$$\mathfrak{C}_\Sigma x_0 = \left\{ w_+ + \mathfrak{W}_{\mathrm{fut}} \;\middle|\; \begin{array}{l} w_+(\cdot) \text{ is the signal part of some stable future} \\ \text{trajectory } (x(\cdot), w_+(\cdot)) \text{ of } \Sigma \text{ with } x(0) = x_0 \end{array} \right\} \tag{5.2}$$

defines a linear contraction $\mathfrak{C}_\Sigma : \mathcal{X} \to \mathcal{H}(\mathfrak{W}_{\mathrm{fut}})$.

Proof. Let $(x(\cdot), w(\cdot))$ be a stable future trajectory of Σ. If $(x_1(\cdot), w_1(\cdot))$ is another stable future trajectory of Σ with the same initial state $x_1(0) = x(0)$, then $w_1(\cdot) - w(\cdot) \in \mathfrak{W}_{\mathrm{fut}}$, and conversely, if $w_1(\cdot) - w(\cdot) \in \mathfrak{W}_{\mathrm{fut}}$, then there exists a stable future trajectory $(x_1(\cdot), w_1(\cdot))$ with $x_1(0) = x(0)$. Thus, the set of all signal parts $w(\cdot)$ of the stable future trajectories $(x(\cdot), w(\cdot))$ of Σ with fixed initial state $x(0) = x_0$ is an equivalence class in $k_-^2(\mathcal{W})/\mathfrak{W}_{\mathrm{fut}}$. By (5.1), the map $\mathfrak{C}_\Sigma$ from x_0 to this equivalence class is a contraction $\mathcal{X} \to \mathcal{H}(\mathfrak{W}_{\mathrm{fut}})$. It is easy to see that this map is linear, and by part (5) of Lemma 2.3, the domain of $\mathfrak{C}_\Sigma$ is all of $\mathcal{X}$. $\square$

Definition 5.3. The contraction $\mathfrak{C}_\Sigma$ in Lemma 5.2 is called the *output map* of Σ.

In our next lemma we need the subspace $\mathfrak{S}_{\mathrm{fut}}^\Sigma$ of $k_+^2(\mathcal{W})$ which is defined as follows:

$$\mathfrak{S}_{\mathrm{fut}}^\Sigma = \{ w(\cdot) \in k_+^2(\mathcal{W}) \mid w + \mathfrak{W}_{\mathrm{fut}} \in \mathcal{R}(\mathfrak{C}_\Sigma) \}. \tag{5.3}$$

We remark that, by Lemma 5.1, it is always true that $\mathfrak{S}_{\mathrm{fut}}^\Sigma \subset \mathcal{K}(\mathfrak{W}_{\mathrm{fut}})$, where $\mathcal{K}(\mathfrak{W}_{\mathrm{fut}})$ is the space defined in (4.13).

Lemma 5.4. *Let $\Sigma = (V; \mathcal{X}; \mathcal{W})$ be a passive s/s system with future behavior $\mathfrak{W}_{\mathrm{fut}}$ and output map $\mathfrak{C}_\Sigma$, and define $\mathfrak{S}_{\mathrm{fut}}^\Sigma$ by* (5.3). *Then every stable future trajectory* $(x(\cdot), w(\cdot))$ *of Σ satisfies*

$$w(\cdot) \in \mathfrak{S}_{\mathrm{fut}}^\Sigma \quad \text{and} \quad \mathfrak{C}_\Sigma x(n) = (S_+^*)^n w + \mathfrak{W}_{\mathrm{fut}}, \quad n \in \mathbb{Z}^+. \tag{5.4}$$

Proof. That $w(\cdot) \in \mathfrak{S}_{\mathrm{fut}}^\Sigma$ follows immediately from (5.3). To get (5.4) we simply shift the trajectory $(x(\cdot), w(\cdot))$ to the left n steps and apply (5.2) with x_0 replaced by $x(n)$. $\square$

Definition 5.5. By an *unobservable future trajectory* of a passive s/s system Σ we mean a (stable) future trajectory of Σ of the type $(x(\cdot), 0)$ (i.e., the signal part is identically zero). The *unobservable* subspace $\mathfrak{U}_\Sigma$ of Σ consists of all the initial states $x(0)$ of all unobservable trajectories of Σ. The system Σ is *observable* if $\mathfrak{U}_\Sigma = \{0\}$.

Lemma 5.6. *The unobservable subspace $\mathfrak{U}_\Sigma$ of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is equal to the null space of its output map $\mathfrak{C}_\Sigma$.*

Proof. It follows directly from Definition 5.5 and Lemma 5.4 that if $x_0 \in \mathfrak{U}_\Sigma$, then $0 \subset \mathfrak{C}_\Sigma x_0$, and hence $\mathfrak{C}_\Sigma x_0$ is the zero element in $\mathcal{H}(\mathfrak{W}_{\mathrm{fut}})$. Conversely, suppose that $x_0 \in \mathcal{N}(\mathfrak{C}_\Sigma)$, i.e., $\mathfrak{C}_\Sigma x_0 = \mathfrak{W}_{\mathrm{fut}}$. By part (5) of Lemma 2.3, there exists a stable future trajectory $(x_1(\cdot), w_1(\cdot))$ of Σ with $x_1(0) = x_0$, and by Lemma 5.4, $w_1(\cdot) \in \mathfrak{C}_\Sigma x_0 = \mathfrak{W}_{\mathrm{fut}}$. Let $(x_2(\cdot), w_1(\cdot))$ be the externally generated future trajectory of Σ whose signal part is $w_1(\cdot)$ (cf. Lemma 2.5), and define $x(\cdot) = x_1(\cdot) - x_2(\cdot)$. Then $(x(\cdot), 0)$ is a stable future trajectory of Σ with $x(0) = x_0$, and hence $x_0 \in \mathfrak{U}_\Sigma$. $\square$

Lemma 5.7. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with output map $\mathfrak{C}_\Sigma$, and define $\mathfrak{S}_{\mathrm{fut}}^\Sigma$ by* (5.3).

(1) *$\mathfrak{S}_{\mathrm{fut}}^\Sigma$ is invariant under S_+^*, i.e., $S_+^* w \in \mathfrak{S}_{\mathrm{fut}}^\Sigma$ whenever $w \in \mathfrak{S}_{\mathrm{fut}}^\Sigma$.*

(2) *To each* $\begin{bmatrix} x_1 \\ x_0 \\ w_0 \end{bmatrix} \in V$, *there exists some* $w \in \mathfrak{S}^{\Sigma}_{\mathrm{fut}}$ *such that*

$$\begin{aligned} \mathfrak{C}_\Sigma x_1 &= S_+^* w + \mathfrak{W}_{\mathrm{fut}}, \\ \mathfrak{C}_\Sigma x_0 &= w + \mathfrak{W}_{\mathrm{fut}}, \\ w_0 &= w(0). \end{aligned} \tag{5.5}$$

(3) *A vector* $\left[\begin{smallmatrix} x_0 \\ w_0 \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ *satisfies the condition* $\begin{bmatrix} x_1 \\ x_0 \\ w_0 \end{bmatrix} \in V$ *for some* $x_1 \in \mathcal{X}$ *if and only if*

$$w_0 = w(0) \quad \textit{for some } w \in \mathfrak{C}_\Sigma x_0. \tag{5.6}$$

Proof. (1) The S_+^*-invariance of $\mathfrak{S}(\Sigma)$ follows from the fact that every left-shifted stable future trajectory of Σ is still a stable future trajectory of Σ.

(2) Let $\begin{bmatrix} x_1 \\ x_0 \\ w_0 \end{bmatrix} \in V$. According to assertion (7) of Lemma 2.3, there exists a stable future trajectory $(x(\cdot), w(\cdot))$ with $x(0) = x_0$, $x(1) = x_1$, and $w(0) = w_0$. In particular, $w \in \mathfrak{S}^{\Sigma}_{\mathrm{fut}}$. By applying (5.4) with $n = 0$ to this trajectory we see that (5.5) holds.

(3) That $\begin{bmatrix} x_1 \\ x_0 \\ w_0 \end{bmatrix} \in V$ implies (5.6) follows from (5.5). Conversely, if (5.6) holds, then there exists some $w(\cdot) \in k_+^2(\mathcal{W})$ with $w(0) = w_0$ such that $w + \mathfrak{W}_{\mathrm{fut}} = \mathfrak{C}_\Sigma x_0$. By definition, this means that there exists some $(x_1(\cdot), w(\cdot))$ with $w(0) = w_0$ which is a stable future trajectory of Σ. By Lemma 5.4, $\mathfrak{C}_\Sigma x_1(0) = w + \mathfrak{W}_{\mathrm{fut}}$. Thus, $\mathfrak{C}_\Sigma(x_0 - x_1(0)) = \mathfrak{W}_{\mathrm{fut}}$, and by Lemma 5.6, $x_0 - x(0)$ belongs to the unobservable subspace of $\mathcal{X}$. This means that there exists a stable future trajectory $(x_2(\cdot), 0)$ of Σ (whose signal part is identically zero) with $x_2(0) = x_0 - x_1(0)$. Define $x(\cdot) = x_1(\cdot) + x_2(\cdot)$. Then $(x(\cdot), w(\cdot))$ is a stable future trajectory of Σ with $x(0) = x_0$ and $w(0) = w_0$, and hence $\begin{bmatrix} x(1) \\ x_0 \\ w_0 \end{bmatrix} \in V$. $\square$

Lemma 5.8. *If the passive s/s system* $\Sigma = (V; \mathcal{X}; \mathcal{W})$ *is observable, then* $(x(\cdot), w(\cdot))$ *is a stable future trajectory of* Σ *if and only if* (5.4) *holds.*

Proof. The necessity of (5.4) follows from Lemma 5.4 and (5.3). Conversely, suppose that (5.4) holds. According to (5.3) there exists at least one stable future trajectory $(x_1(\cdot), w(\cdot))$ of Σ, and by Lemma 5.4, (5.4) holds with $x(\cdot)$ replaced by $x_1(\cdot)$. By Lemma 5.6 and the observability assumption on Σ, $\mathfrak{C}_\Sigma$ is injective, and hence (5.4) implies that $x(n) = x_1(n)$ for all $n \in \mathbb{Z}^+$. This implies that $(x(\cdot), w(\cdot))$ is a stable future trajectory of Σ. $\square$

Lemma 5.9. *Let* $\Sigma = (V; \mathcal{X}, \mathcal{W})$ *be a passive s/s system with output map* $\mathfrak{C}_\Sigma$. *Then* $(x(\cdot), w(\cdot))$ *is a stable future trajectory of* Σ *if and only if* $x(\cdot) = x_1(\cdot) + x_2(\cdot)$, *where* $(x_1(\cdot), 0)$ *is an unobservable future trajectory of* Σ *and* $(x_2(\cdot), w(\cdot))$ *is a stable future trajectory of* Σ *with* $x_2(0) \in (\mathcal{N}(\mathfrak{C}_\Sigma))^\perp$. *This decomposition is unique, and* (5.4) *also holds with* $x(\cdot)$ *replaced by* $x_2(\cdot)$.

Proof. Trivially, if $x(\cdot)$ has a decomposition of the type described in the lemma, then $(x(\cdot), w(\cdot))$ is a stable future trajectory of Σ.

Conversely, let $(x(\cdot), w(\cdot))$ be a stable future trajectory of Σ. Define $x_1(0) = P_{\mathfrak{U}_\Sigma} x(0)$ and $x_2(0) = P_{\mathfrak{U}_\Sigma^\perp} x(0)$. Then $x(0) = x_1(0) + x_2(0)$ and $x_1(0) \in \mathfrak{U}_\Sigma$. The latter condition implies that $x_1(0)$ is the initial state of some unobservable trajectory $(x_1(\cdot), 0)$ of Σ. Define $x_2(\cdot) = x(\cdot) - x_1(\cdot)$. Then $(x_2(\cdot), w(\cdot))$ is a stable future trajectory of Σ and $x(\cdot) = x_1(\cdot) + x_2(\cdot)$. That (5.4) also holds $x(\cdot)$ replaced by $x_2(\cdot)$ follows from the fact that $(x_2(\cdot), w(\cdot))$ is a stable future trajectory of Σ. □

The input map $\mathfrak{B}_\Sigma$

We now proceed to the construction of the input map $\mathfrak{B}_\Sigma$ of a passive s/s system Σ.

Lemma 5.10. *Let $\Sigma = (V; \mathcal{X}; \mathcal{W})$ be a passive s/s system with past behavior $\mathfrak{W}_{\mathrm{past}}$. Then there exists a unique linear contraction $\mathfrak{B}_\Sigma : \mathcal{H}(\mathfrak{W}_{\mathrm{past}}^{[\perp]}) \to \mathcal{X}$ whose restriction to $\mathcal{H}^0(\mathfrak{W}_{\mathrm{past}}^{[\perp]})$ is given by*

$$\mathfrak{B}_\Sigma\bigl(w_- + \mathfrak{W}_{\mathrm{past}}^{[\perp]}\bigr) = x(0), \quad w_-(\cdot) \in \mathfrak{W}_{\mathrm{past}}, \tag{5.7}$$

where $(x(\cdot), w_-(\cdot))$ is the unique stable externally generated past trajectory of Σ whose signal part is $w_-(\cdot)$ (cf. Lemma 2.5).

Proof. Let $w(\cdot) \in \mathfrak{W}_{\mathrm{past}}$, and let $(x(\cdot), w(\cdot))$ be the externally generated stable past trajectory of Σ with signal part $w(\cdot)$. Then by (2.8) and (4.19)

$$\bigl\|x(0)\bigr\|_{\mathcal{X}}^2 \leqslant \bigl[w(\cdot), w(\cdot)\bigr]_{k_-^2(\mathcal{W})} = \bigl\|w + \mathfrak{W}_{\mathrm{past}}^{[\perp]}\bigr\|_{\mathcal{H}(\mathfrak{W}_{\mathrm{past}}^{[\perp]})}^2.$$

This implies that the mapping $w + \mathfrak{W}_{\mathrm{past}}^{[\perp]} \to x(0)$ is a linear contraction $\mathcal{H}^0(\mathfrak{W}_{\mathrm{past}}^{[\perp]}) \to \mathcal{X}$. Since $\mathcal{H}^0(\mathfrak{W}_{\mathrm{past}}^{[\perp]})$ is dense in $\mathcal{H}(\mathfrak{W}_{\mathrm{past}}^{[\perp]})$, this mapping has a unique extension to a linear contraction $\mathfrak{B}_\Sigma : \mathcal{H}(\mathfrak{W}_{\mathrm{past}}^{[\perp]}) \to \mathcal{X}$. □

Definition 5.11. The contraction $\mathfrak{B}_\Sigma$ in Lemma 5.10 is called the *input map* of Σ.

Lemma 5.12. *Let $\Sigma = (V; \mathcal{X}; \mathcal{W})$ be a passive s/s system with past behavior $\mathfrak{W}_{\mathrm{past}}$, future behavior $\mathfrak{W}_{\mathrm{fut}}$, input map $\mathfrak{B}_\Sigma$, and output map $\mathfrak{C}_\Sigma$. Then $(x(\cdot), w(\cdot))$ is an externally generated stable past trajectory of Σ if and only if*

$$w \in \mathfrak{W}_{\mathrm{past}} \quad \textit{and} \quad x(n) = \mathfrak{B}_\Sigma\bigl(S_-^{-n} w + \mathfrak{W}_{\mathrm{past}}^{[\perp]}\bigr), \quad n \leqslant 0, \tag{5.8}$$

and $(x(\cdot), w(\cdot))$ is an externally generated stable full trajectory of Σ if and only if

$$w \in \mathfrak{W}_{\mathrm{full}} \quad \textit{and} \quad x(n) = \mathfrak{B}_\Sigma\bigl(\pi_- S^{-n} w + \mathfrak{W}_{\mathrm{past}}^{[\perp]}\bigr), \quad n \in \mathbb{Z}. \tag{5.9}$$

In the latter case we have, in addition,

$$\mathfrak{C}_\Sigma x(n) = \pi_+ S^{-n} w + \mathfrak{W}_{\mathrm{fut}}, \quad n \in \mathbb{Z}. \tag{5.10}$$

Proof. The proof of the claim about past trajectories is an easy modification of the proof of the first claim about full trajectories, so let us only prove the two claims about the full trajectories.

Let $(x(\cdot), w(\cdot))$ be an externally generated stable full trajectory of Σ. Then $w(\cdot) \in \mathfrak{W}_{\text{full}}$, and (5.7) implies that (5.9) holds with $n = 0$. By shifting the trajectory to the left or right $|n|$ steps and applying (5.7) to the shifted trajectory we get (5.8) for all values of $n \in \mathbb{Z}$.

Conversely, let $w(\cdot) \in \mathfrak{W}_{\text{full}}$. Then there exists a sequence $x(\cdot)$ such that $(x(\cdot), w(\cdot))$ is an externally generated stable full trajectory of Σ, and by the first part of the proof, the sequence $x(\cdot)$ is given by (5.9).

That also (5.10) holds follows from Lemma 5.4 and the fact that the restriction to $\mathbb{Z}^+$ of any left- or right-shifted externally generated stable full trajectory of Σ is a stable future trajectory of Σ. □

Definition 5.13. By the *finite time exactly reachable subspace* of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ we mean the set

$$\left\{x_0 \in \mathcal{X} \;\middle|\; \begin{matrix} x_0 = x(0) \text{ for some (stable) past} \\ \text{trajectory of } \Sigma \text{ with finite support} \end{matrix}\right\},$$

by the *infinite time exactly reachable subspace* of Σ we mean the set

$$\left\{x_0 \in \mathcal{X} \;\middle|\; \begin{matrix} x_0 = x(0) \text{ for some stable externally} \\ \text{generated past trajectory of } \Sigma \end{matrix}\right\},$$

and by the $\mathcal{H}(\mathfrak{W}_{\text{past}}^{[\perp]})$-*exactly reachable subspace* of Σ we mean the range of the input map $\mathfrak{B}_\Sigma$ of Σ. The system Σ is *exactly reachable in one of the above senses* if the corresponding exactly reachable subspace is all of $\mathcal{X}$. The closure of the first of these three subspaces is called the (*approximately*) *reachable subspace*. Finally, Σ is *approximately reachable* or *controllable* if the approximately reachable subspace is all of $\mathcal{X}$.

Lemma 5.14. *All the different types of exactly reachable subspaces in Definition* 5.13 *have the same closure, equal to the approximately reachable subspace.*

Proof. The three different types of exactly reachable subspaces defined in Definition 5.13 are (in the order that they appear) the range of the restriction of $\mathfrak{B}_\Sigma$ to the space $\mathcal{H}_0^0(\mathfrak{W}_{\text{past}}^{[\perp]})$ defined in Lemma 4.2, the range of the restriction of $\mathfrak{B}_\Sigma$ to the space $\mathcal{H}_0(\mathfrak{W}_{\text{past}}^{[\perp]})$, and the full range of $\mathfrak{B}_\Sigma$. That these three subspaces have the same closure follows from the fact that when one restricts the bounded linear operator $\mathfrak{B}_\Sigma$ to a dense subset of its domain, then the closure of its range remains the same. □

Lemma 5.15. *If Σ is a passive forward conservative s/s system, then the input map $\mathfrak{B}_\Sigma$ of Σ is an isometry. If, in addition, Σ is controllable, then $\mathfrak{B}_\Sigma$ is unitary.*

Proof. That $\mathfrak{B}_\Sigma$ is an isometry follows from the fact that we have equality in (2.8) whenever Σ is forward conservative. In particular, $\mathcal{R}(\mathfrak{B}_\Sigma)$ is closed. If, in addition, Σ is controllable, then $\mathcal{R}(\mathfrak{B}_\Sigma)$ is dense in $\mathcal{X}$, and hence equal to $\mathcal{X}$. □

Lemma 5.16. *In the setting of Lemma* 5.12, *the subspace*

$$\mathring{V} := \left\{ \begin{bmatrix} x(0) \\ x(-1) \\ w(-1) \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \;\middle|\; \begin{matrix} \big(x(\cdot), w(\cdot)\big) \textit{ is a stable externally} \\ \textit{generated past trajectory of } \Sigma \end{matrix} \right\} \tag{5.11}$$

of V is dense in V if and only if the system Σ is controllable, and it is equal to V if and only if Σ is infinite time exactly reachable.

Proof. Suppose that $\mathring{V}$ is dense in V. This implies that the infinite time exactly reachable subspace is dense in $\mathcal{X}$, and by Lemma 5.14, this implies that Σ is controllable.

Conversely, suppose that Σ is controllable. By Lemma 2.3, every stable externally generated past trajectory of Σ can be extended to a stable externally generated full trajectory of Σ, and Eq. (5.11) can be rewritten in the equivalent form (where we have shifted the extended trajectory one step to the left)

$$\mathring{V} := \left\{ \begin{bmatrix} x(1) \\ x(0) \\ w(0) \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \;\middle|\; \begin{matrix} \big(x(\cdot), w(\cdot)\big) \text{ is a stable externally} \\ \text{generated full trajectory of } \Sigma \end{matrix} \right\}. \tag{5.12}$$

Let $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ be a fundamental decomposition of $\mathcal{W}$. This induces a fundamental decomposition

$$\mathfrak{K} := \begin{bmatrix} -\mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} = \begin{bmatrix} -\mathcal{X} \\ 0 \\ -\mathcal{Y} \end{bmatrix} [\dot{+}] \begin{bmatrix} 0 \\ \mathcal{X} \\ \mathcal{U} \end{bmatrix}$$

of the node space $\mathfrak{K}$. We claim that the orthogonal projection of $\mathring{V}$ onto the uniformly positive subspace $\begin{bmatrix} 0 \\ \mathcal{X} \\ \mathcal{U} \end{bmatrix}$ in this decomposition is dense in $\begin{bmatrix} 0 \\ \mathcal{X} \\ \mathcal{U} \end{bmatrix}$. This projection is equal to

$$\left\{ \begin{bmatrix} 0 \\ x(0) \\ P_{\mathcal{U}} w(0) \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \;\middle|\; \begin{matrix} \big(x(\cdot), w(\cdot)\big) \text{ is a stable externally} \\ \text{generated full trajectory of } \Sigma \end{matrix} \right\}.$$

The above set does not change if we replace the trajectory $(x(\cdot), w(\cdot))$ in the parametrization above by $(x(\cdot), w(\cdot)) = (x_1(\cdot) + x_2(\cdot), w_1(\cdot) + w_2(\cdot))$, where $(x_1(\cdot), w_1(\cdot))$ is a stable externally generated full trajectory of Σ and $(x_2(\cdot), w_2(\cdot))$ is a stable externally generated future trajectory of Σ (since the result is still a stable full externally generated trajectory of Σ). By part (4) of Lemma 2.3, if one first fixes $(x_1(\cdot), w_1(\cdot))$, and hence fixes $x(0)$, then it is still possible to choose $(x_2(\cdot), w_2(\cdot))$ in such a way that $P_{\mathcal{U}} w(0) = P_{\mathcal{U}}(w_1(0) + w_2(0))$ is an arbitrary vector in $\mathcal{U}$. This implies that the orthogonal projection of $\mathring{V}$ onto $\begin{bmatrix} 0 \\ \mathcal{X} \\ \mathcal{U} \end{bmatrix}$ is $\begin{bmatrix} 0 \\ \mathcal{X}_0 \\ \mathcal{U} \end{bmatrix}$, where $\mathcal{X}_0$ is the infinite-time exactly reachable subspace of Σ. This is a dense subspace of $\begin{bmatrix} 0 \\ \mathcal{X} \\ \mathcal{U} \end{bmatrix}$, as claimed.

Since V is maximal nonnegative, it has a graph representation of the form

$$V = \left\{ \begin{bmatrix} Ax + Bu \\ x \\ Cx + Du \end{bmatrix} \;\middle|\; x \in \mathcal{X} \text{ and } u \in \mathcal{U} \right\}, \tag{5.13}$$

for some contraction $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$. The subspace $\mathring{V}$ is equal to

$$\mathring{V} = \left\{ \begin{bmatrix} Ax + Bu \\ x \\ Cx + Du \end{bmatrix} \,\middle|\, x \in \mathcal{X}_0 \text{ and } u \in \mathcal{U} \right\}. \tag{5.14}$$

Since $\left[\begin{smallmatrix} \mathcal{X}_0 \\ \mathcal{U} \end{smallmatrix}\right]$ is dense in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$, this implies that $\mathring{V}$ is dense in V. It is equal to V if and only if $\mathcal{X}_0 = \mathcal{X}$, i.e., if Σ is infinite time exactly reachable. □

The adjoints of $\mathfrak{C}_\Sigma$ and $\mathfrak{B}_\Sigma$

The rest of this section is devoted to the study of the adjoints of the input and output maps of a passive s/s system.

Lemma 5.17. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with past and future behaviors $\mathfrak{W}_{\mathrm{past}}$ and $\mathfrak{W}_{\mathrm{fut}}$, respectively, and let $\Sigma^\dagger = (V^{[\perp]}; \mathcal{X}, \mathcal{W})$ be the anti-passive dual of Σ with past and future behaviors $\mathfrak{W}_{\mathrm{past}}^{[\perp]}$ and $\mathfrak{W}_{\mathrm{fut}}^{[\perp]}$, respectively.*

(1) *There exists a unique contraction $\mathfrak{B}_{\Sigma^\dagger} : \mathcal{H}(\mathfrak{W}_{\mathrm{fut}}) \to \mathcal{X}$ such that $(x^\dagger(\cdot), w^\dagger(\cdot))$ is an externally generated stable future trajectory of $\Sigma^\dagger$ if and only if $w^\dagger \in \mathfrak{W}_{\mathrm{fut}}^\dagger$ and*

$$x^\dagger(n) = \mathfrak{B}_{\Sigma^\dagger}\left(S_+^*\right)^n w^\dagger, \quad n \in \mathbb{Z}^+. \tag{5.15}$$

(2) *There exists a unique contraction $\mathfrak{C}_{\Sigma^\dagger} : \mathcal{X} \to \mathcal{H}(\mathfrak{W}_{\mathrm{past}}^{[\perp]})$ satisfying*

$$\mathfrak{C}_{\Sigma^\dagger} x(-n) = (S_-)^n w^\dagger + \mathfrak{W}_{\mathrm{past}}^{[\perp]} \tag{5.16}$$

for every stable past trajectory $(x^\dagger(\cdot), w^\dagger(\cdot))$ of $\Sigma^\dagger$.

Proof. Claim (1) is the anti-passive version of Lemma 5.12, and claim (2) is the anti-passive version of Lemma 5.4. They can be proved by either repeating the proofs of these two lemmas, or by applying Lemmas 5.12 and 5.4 to the passive dual Σ_* of Σ. □

Definition 5.18. The contractions $\mathfrak{B}_{\Sigma^\dagger}$ and $\mathfrak{C}_{\Sigma^\dagger}$ are called the input and output maps of $\Sigma^\dagger$, respectively.

Lemma 5.19. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with input map $\mathfrak{B}_\Sigma$ and output map $\mathfrak{C}_\Sigma$, and let $\Sigma^\dagger$ be the anti-passive dual of Σ, with the input map $\mathfrak{B}_{\Sigma^\dagger}$ and output map $\mathfrak{C}_{\Sigma^\dagger}$. Then $\mathfrak{B}_{\Sigma^\dagger} = \mathfrak{C}_\Sigma^*$ and $\mathfrak{C}_{\Sigma^\dagger} = \mathfrak{B}_\Sigma^*$.*

Proof. Let $\mathfrak{W}_{\mathrm{past}}$ and $\mathfrak{W}_{\mathrm{fut}}$ be the past and future behaviors of Σ, respectively. Let $(x(\cdot), w(\cdot))$ be an externally generated past trajectory of Σ, and let $(x^\dagger(\cdot), w^\dagger(\cdot))$ be a stable past trajectory of $\Sigma^\dagger$. Then, by (3.3) and (5.8),

$$\begin{aligned}\big(\mathfrak{B}_\Sigma\big(w+\mathfrak{W}_{\mathrm{past}}^{[\perp]}\big), x^\dagger(0)\big)_{\mathcal{X}} &= \big(x(0), x^\dagger(0)\big)_{\mathcal{X}} \\ &= \big[w(\cdot), w^\dagger(\cdot)\big]_{k_-^2(\mathcal{W})} \\ &= \big(w(\cdot)+\mathfrak{W}_{\mathrm{past}}^{[\perp]}, w^\dagger(\cdot)+\mathfrak{W}_{\mathrm{past}}^{[\perp]}\big)_{\mathcal{H}(\mathfrak{W}_{\mathrm{past}}^{[\perp]})} \\ &= \big(w(\cdot)+\mathfrak{W}_{\mathrm{past}}^{[\perp]}, \mathfrak{C}_{\Sigma^\dagger} x^\dagger(0)\big)_{\mathcal{H}(\mathfrak{W}_{\mathrm{past}}^{[\perp]})}.\end{aligned}$$

This implies that $(\mathfrak{B}_\Sigma h, x^\dagger)_{\mathcal{X}} = (x, \mathfrak{C}_{\Sigma^\dagger} x^\dagger)_{\mathcal{H}(\mathfrak{W}_{\mathrm{past}}^{[\perp]})}$ for every $h \in \mathcal{H}^0(\mathfrak{W}_{\mathrm{past}}^{[\perp]})$ and every $x^\dagger \in \mathcal{X}$. Since $\mathcal{H}^0(\mathfrak{W}_{\mathrm{past}}^{[\perp]})$ is dense in $\mathcal{H}(\mathfrak{W}_{\mathrm{past}}^{[\perp]})$, this implies that $\mathfrak{B}_\Sigma^* = \mathfrak{C}_{\Sigma^\dagger}$.

The proof of the fact that $\mathfrak{C}_{\Sigma^\dagger} = \mathfrak{B}_\Sigma^*$ is similar to the one above, and it is left to the reader (start by taking a stable future trajectory $(x(\cdot), w(\cdot))$ of Σ and a stable externally generated future trajectory $(x^\dagger(\cdot), w^\dagger(\cdot))$ of $\Sigma^\dagger$). □

Lemma 5.20. *If Σ is a backward conservative passive s/s system, then the output map $\mathfrak{C}_\Sigma$ of Σ is a co-isometry. If, in addition, Σ is observable, then $\mathfrak{C}_\Sigma$ is unitary.*

Proof. The first claim follows from the fact that if Σ is backward conservative, then the anti-passive dual $\Sigma^\dagger$ is forward conservative, and hence its input map $\mathfrak{B}_{\Sigma^\dagger} = \mathfrak{C}_\Sigma^*$ is an isometry. The second claim follows from the first claim since $\mathfrak{C}_\Sigma$ is injective iff Σ is observable. □

6. The past/future map of a passive full behavior

We begin by constructing the past/future map of a given passive full behavior $\mathfrak{W}$, and then investigate what can be said about this map in the case where $\mathfrak{W}$ is the full behavior of a passive s/s system Σ.

Lemma 6.1. *Let $\mathfrak{W}$ be a passive full behavior on $\mathcal{W}$ with the corresponding passive past behavior $\mathfrak{W}_- = \pi_-\mathfrak{W}$ and passive future behavior $\mathfrak{W}_+ = \mathfrak{W} \cap k_+^2(\mathcal{W})$. Then there exists a unique contraction $\Gamma_{\mathfrak{W}} : \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \to \mathcal{H}(\mathfrak{W}_+)$ satisfying*

$$\Gamma_{\mathfrak{W}}\big(\pi_- w + \mathfrak{W}_-^{[\perp]}\big) = \pi_+ w + \mathfrak{W}_+, \quad w \in \mathfrak{W}. \tag{6.1}$$

Proof. Since $\mathfrak{W}$ is nonnegative in $k^2(\mathcal{W})$ and $\mathfrak{W}_+ = \mathfrak{W} \cap k_+^2(\mathcal{W})$, we have for all $w \in \mathfrak{W}$ and all $z \in \mathfrak{W}_+$,

$$0 \leqslant [w+z, w+z]_{k^2(\mathcal{W})} = [\pi_- w, \pi_- w]_{k_-^2(\mathcal{W})} + [\pi_+ w + z, \pi_+ w + z]_{k_+^2(\mathcal{W})}.$$

Consequently,

$$-[\pi_+ w + z, \pi_+ w + z]_{k_+^2(\mathcal{W})} \leqslant [\pi_- w, \pi_- w]_{k_-^2(\mathcal{W})} = \big\|\pi_- w + \mathfrak{W}_-^{[\perp]}\big\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})}$$

for every $w \in \mathfrak{W}$ and every $z \in \mathfrak{W}_+$. This implies that $\pi_+ w + \mathfrak{W}_+ \in \mathcal{H}(\mathfrak{W}_+)$, and that

$$\|\pi_+ w + \mathfrak{W}_+\|_{\mathcal{H}(\mathfrak{W}_+)} \leqslant \big\|\pi_- w + \mathfrak{W}_-^{[\perp]}\big\|_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})}. \tag{6.2}$$

If both $w_1 \in \mathfrak{W}$ and $w_2 \in \mathfrak{W}$ and $\pi_-(w_1 - w_2) \in \mathfrak{W}_-^{[\perp]}$, then by the above argument, $\pi_+(w_1 - w_2) \in \mathfrak{W}_+$ and

$$\|\pi_+ w_1 - \pi_+ w_2 + \mathfrak{W}_+\|_{\mathcal{H}(\mathfrak{W}_+)} \leqslant \left\|\pi_-(w_1 - w_2) + \mathfrak{W}_-^{[\perp]}\right\|_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} = 0.$$

Consequently, $\pi_+ w_1 - \pi_+ w_2 \in \mathfrak{W}_+$. Thus, formula (6.1) defines a (unique) linear contraction $\mathcal{H}^0(\mathfrak{W}_-^{[\perp]}) \to \mathcal{H}(\mathfrak{W}_+)$, and since $\mathcal{H}^0(\mathfrak{W}_-^{[\perp]})$ is dense in $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$, it has a unique extension to a linear contraction $\Gamma_{\mathfrak{W}} : \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \to \mathcal{H}(\mathfrak{W}_+)$. □

Definition 6.2. The contraction $\Gamma_{\mathfrak{W}} : \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \to \mathcal{H}(\mathfrak{W}_+)$ in Lemma 6.1 is called the *past/future map* of the full behavior $\mathfrak{W}$. If $\mathfrak{W}$ is the full behavior of a passive s/s system Σ, then we also call $\Gamma_{\mathfrak{W}}$ the past/future map of Σ and denote it by Γ_Σ.

Lemma 6.3. *The past/future map Γ_Σ of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ factors into the product*

$$\Gamma_\Sigma = \mathfrak{C}_\Sigma \mathfrak{B}_\Sigma \tag{6.3}$$

of the input map $\mathfrak{B}_\Sigma$ and the output map $\mathfrak{C}_\Sigma$ of Σ. In particular, if Σ_i, $i = 1, 2$, are two externally equivalent passive s/s systems, with input maps $\mathfrak{B}_{\Sigma_i}$ and output maps $\mathfrak{C}_{\Sigma_i}$, then $\mathfrak{C}_{\Sigma_1}\mathfrak{B}_{\Sigma_1} = \mathfrak{C}_{\Sigma_2}\mathfrak{B}_{\Sigma_2}$.

Proof. Let $(x(\cdot), w(\cdot))$ be an externally generated stable full trajectory of Σ. Then the restriction of $(x(\cdot), w(\cdot))$ to $\mathbb{Z}^-$ is an externally generated stable past trajectory and the restriction of $(x(\cdot), w(\cdot))$ to $\mathbb{Z}^+$ is a stable future trajectory of Σ. Thus, by (5.8), $x(0) = \mathfrak{B}_\Sigma \pi_- w$ and by (5.4), $\mathfrak{C}_\Sigma x(0) = \pi_+ w + \mathfrak{W}_{\mathrm{fut}}$. Thus, the two contractions $\Gamma_{\mathfrak{W}}$ and $\mathfrak{C}_\Sigma \mathfrak{B}_\Sigma$ coincide on the dense subspace $\mathcal{H}^0(\mathfrak{W}_-^{[\perp]})$ of $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$, and hence on all of $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$. If the systems Σ_i, $i = 1, 2$, are externally equivalent, then they have the same full behavior $\mathfrak{W}$ and hence the same past/future map $\Gamma_{\mathfrak{W}}$. Thus $\mathfrak{C}_{\Sigma_1}\mathfrak{B}_{\Sigma_1} = \Gamma_{\mathfrak{W}} = \mathfrak{C}_{\Sigma_2}\mathfrak{B}_{\Sigma_2}$. □

Lemma 6.4. *Let $\mathfrak{W}$ be a full behavior with the corresponding past behavior $\mathfrak{W}_-$ and future behavior $\mathfrak{W}_+$. Then there is a unique contraction $\Gamma_{\mathfrak{W}^{[\perp]}} : \mathcal{H}(\mathfrak{W}_+) \to \mathcal{H}(\mathfrak{W}_-^{[\perp]})$ satisfying*

$$\Gamma_{\mathfrak{W}^{[\perp]}}\big(\pi_+ w^\dagger + \mathfrak{W}_+\big) = \pi_- w^\dagger + \mathfrak{W}_-^{[\perp]}, \quad w^\dagger \in \mathfrak{W}^{[\perp]}. \tag{6.4}$$

Proof. The proof is the same as the proof of Lemma 6.1 with the following replacements: We interchange $\pi_- \leftrightarrow \pi_+$, $\mathfrak{W} \leftrightarrow -\mathfrak{W}^{[\perp]}$, $\mathfrak{W}_+ \leftrightarrow -\mathfrak{W}_-^{[\perp]} = -\mathfrak{W}^{[\perp]} \cap k_-^2(\mathcal{W})$ and $\mathfrak{W}_- \leftrightarrow -\mathfrak{W}^{[\perp]} \cap k_+^2(\mathcal{W})$. □

Definition 6.5. The contraction $\Gamma_{\mathfrak{W}^{[\perp]}} : \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \to \mathcal{H}(\mathfrak{W}_+)$ in Lemma 6.1 is called the *future/past map* of the anti-passive full behavior $\mathfrak{W}^{[\perp]}$. If $\mathfrak{W}^{[\perp]}$ is the full behavior of a passive anti-causal s/s system $\Sigma^\dagger$, then we also call $\Gamma_{\mathfrak{W}^{[\perp]}}$ the future/past map of $\Sigma^\dagger$ and denote it by $\Gamma_{\Sigma^\dagger}$.

Lemma 6.6. *The future/past map $\Gamma_{\Sigma^\dagger}$ of the anti-passive full behavior $\mathfrak{W}_{\mathrm{full}}^{[\perp]}$ induced by a anti-passive reflected s/s system $\Sigma^\dagger$ factors into the product*

$$\Gamma_{\Sigma^\dagger} = \mathfrak{C}_{\Sigma^\dagger}\mathfrak{B}_{\Sigma^\dagger} \tag{6.5}$$

of the input map $\mathfrak{B}_{\Sigma^\dagger}$ of $\Sigma^\dagger$ and the output map $\mathfrak{C}_{\Sigma^\dagger}$ of $\Sigma^\dagger$.

Proof. The proof is analogous to the proof of Lemma 6.3. □

Lemma 6.7. *The adjoint of the past/future map $\Gamma_{\mathfrak{W}}$ of the full behavior $\mathfrak{W}$ is the future/past map $\Gamma_{\mathfrak{W}^{[\perp]}}$ of the dual behavior $\mathfrak{W}^{[\perp]}$.*

Proof. This follows from Lemmas 6.3, 5.19, and 6.6. □

Lemma 6.8. *Let $\mathfrak{W}$ be a passive full behavior with the corresponding passive past behavior $\mathfrak{W}_- = \pi_-\mathfrak{W}$ and passive future behavior $\mathfrak{W}_+ = \mathfrak{W} \cap k_+^2(\mathcal{W})$. Let $w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$, $w_+ \in \mathcal{K}(\mathfrak{W}_+)$, and suppose that*

$$w_+ + \mathfrak{W}_+ = \Gamma_{\mathfrak{W}}\big(w_- + \mathfrak{W}_-^{[\perp]}\big). \tag{6.6}$$

Denote $w := w_- + w_+$ Then, for all $n \in \mathbb{Z}^+$, $\pi_-S^{-n}w \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$, $\pi_+S^{-n}w \in \mathcal{K}(\mathfrak{W}_+)$,

$$\pi_+S^{-n}w + \mathfrak{W}_+ = \Gamma_{\mathfrak{W}}\big(\pi_-S^{-n}w + \mathfrak{W}_-^{[\perp]}\big), \quad n \in \mathbb{Z}^+, \tag{6.7}$$

$$\begin{aligned}\big\|\pi_-S^{-n-1}w + \mathfrak{W}_-^{[\perp]}\big\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} &= \big\|\pi_-S^{-n}w + \mathfrak{W}_-^{[\perp]}\big\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} \\ &\quad + \big[w_+(n), w_+(n)\big]_{\mathcal{W}}, \quad n \in \mathbb{Z}^+.\end{aligned} \tag{6.8}$$

Moreover, there exists a sequence $w^k \in \mathfrak{W}$ such that

$$\pi_+S^{-n}w^k + \mathfrak{W}_+ \to \pi_+S^{-n}w_+ + \mathfrak{W}_+ \quad \text{in } \mathcal{H}(\mathfrak{W}_+),\ n \in \mathbb{Z}^+, \tag{6.9}$$

$$\pi_-S^{-n}w^k + \mathfrak{W}_-^{[\perp]} \to \pi_-S^{-n}w + \mathfrak{W}_-^{[\perp]} \quad \text{in } \mathcal{H}\big(\mathfrak{W}_-^{[\perp]}\big),\ n \in \mathbb{Z}^+, \tag{6.10}$$

$$\pi_+w^k \to w_+ \quad \text{in } k_+^2(\mathcal{W}), \tag{6.11}$$

as $n \to \infty$, where the convergence in (6.9) *and* (6.10) *is uniform in n.*

Proof. *Step* 1: *Proofs of* (6.9)–(6.11) *with $n = 0$.* Since $\mathcal{H}^0(\mathfrak{W}_-^{[\perp]})$ is dense in $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$, there exists a sequence $w_-^k \in \mathfrak{W}_-$ such that $w_-^k + \mathfrak{W}_-^{[\perp]} \to w_- + \mathfrak{W}_-^{[\perp]}$ in $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ as $k \to \infty$. As $\mathfrak{W}_- = \pi_-\mathfrak{W}$, it is possible to extend each w_-^k to a function $w^k \in \mathfrak{W}$, i.e., $w_-^k = \pi_-w^k$. Then (6.10) holds with $n = 0$ for this sequence w^k. By the definition of Γ_Σ,

$$\pi_+w^k(\cdot) + \mathfrak{W}_+ = \Gamma_{\mathfrak{W}}\big(\pi_-w^k(\cdot) + \mathfrak{W}_-^{[\perp]}\big), \quad k \in \mathbb{Z}^+. \tag{6.12}$$

Since $\Gamma_{\mathfrak{W}} \in \mathcal{B}(\mathcal{H}(\mathfrak{W}_-^{[\perp]}); \mathcal{H}(\mathfrak{W}_+))$, this implies that

$$\pi_+ w^k(\cdot) + \mathfrak{W}_+ \to \Gamma_{\mathfrak{W}}\big(w_- + \mathfrak{W}_-^{[\perp]}\big) \quad \text{in } \mathcal{H}(\mathfrak{W}_+).$$

This together with (6.6) gives (6.9) with $n = 0$. Then, by Theorem 4.1, there exists a sequence $z_+^k \in \mathfrak{W}_+$ such that $\pi_+ w^k + z_+^k \to w_+$ in $k_+^2(\mathcal{W})$. If we replace w^k by $\tilde{w}^k = w^k + z_+^k$, then (6.9) and (6.10) remain valid, and also (6.11) holds.

Step 2: *Proof of* (6.8) *with* $n = 0$. Let w^k be a sequence satisfying (6.9)–(6.11) with $n = 0$. Then $S^{-1}w^k \in \mathfrak{W}$, and consequently $\pi_- S^{-1} w^k \in \mathfrak{W}_-$. By Lemma 4.6, $S_-\pi_- S^{-1}w^k \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$ and

$$\begin{aligned}\big\|S_-\pi_- S^{-1}w^k + \mathfrak{W}_-^{[\perp]}\big\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} &= \big\|\pi_- S^{-1}w^k + \mathfrak{W}_-^{[\perp]}\big\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} \\ &\quad - \big[\big(\pi_- S^{-1}w^k\big)(-1), \big(\pi_- S^{-1}w^k\big)(-1)\big]_{\mathcal{W}}.\end{aligned}$$

Here $S_-\pi_- S^{-1}w^k = \pi_- w^k$ and $(\pi_- S^{-1}w^k)(-1) = w^k(0) = w_+(0)$ where $w_+ = \pi_+ w$. Consequently,

$$\big\|\pi_- S^{-1}w^k + \mathfrak{W}_-^{[\perp]}\big\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} = \big\|\pi_- w^k + \mathfrak{W}_-^{[\perp]}\big\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} + \big[w^k(0), w^k(0)\big]_{\mathcal{W}}. \tag{6.13}$$

By (6.10) with $n = 0$ and by (6.11), the right-hand side of this identity tends to the right-hand side of (6.8) with $n = 0$, so to prove (6.8) with $n = 0$ it suffices to show that $\pi_- S^{-1}w^k + \mathfrak{W}_-^{[\perp]} \to \pi_- S^{-1}w + \mathfrak{W}_-^{[\perp]}$ in $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ as $k \to \infty$. We begin by showing that $\lim_{k\to\infty} \pi_- S^{-1}w^k + \mathfrak{W}_-^{[\perp]}$ exists in $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$. The identity (6.13) also holds with w^k replaced by $w^k - w^\ell$ for all $k, \ell \in \mathbb{Z}^+$. From this and conditions (6.9) and (6.10) follows that $\pi_- S^{-1}w^k + \mathfrak{W}_-^{[\perp]}$ is a Cauchy sequence in $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$, and hence $\pi_- S^{-1}w^k + \mathfrak{W}_-^{[\perp]} \to h_1$ in $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ for some $h_1 \in \mathcal{H}(\mathfrak{W}_-^{[\perp]})$. We still have to show that $h_1 = \pi_- S^{-1}w + \mathfrak{W}_-^{[\perp]}$. By Theorem 4.4, there exists a sequence $z_-^k \in \mathfrak{W}_-^{[\perp]}$ such that $\pi_- w^k + z_-^k \to w_-$ in $k_-^2(\mathcal{W})$. Then, by (6.11),

$$w^k + z_-^k = \pi_-\big(w^k + z_-^k\big) + \pi_+ w^k \to w_- + \pi_+ w = w$$

and

$$\pi_\pm S^{-1}\big(w^k + z_-^k\big) \to \pi_\pm S^{-1}w \quad \text{in } k_\pm^2(\mathcal{W}) \tag{6.14}$$

as $k \to \infty$. Moreover,

$$\pi_- S^{-1}\big(w^k + z_-^k\big) + \mathfrak{W}_-^{[\perp]} = \pi_- S^{-1}w^k + \mathfrak{W}_-^{[\perp]} \to h_1 \quad \text{in } \mathcal{H}\big(\mathfrak{W}_-^{[\perp]}\big) \tag{6.15}$$

as $k \to \infty$. By Theorem 4.4, the restriction of the quotient map $w(\cdot) \to w(\cdot) + \mathfrak{W}_-^{[\perp]}$ to $\mathcal{K}(\mathfrak{W}_-^{[\perp]})$ is a closed operator $k_-^2(\mathcal{W}) \to \mathcal{H}(\mathfrak{W}_-^{[\perp]})$, and thus $\pi_- S^{-1}w + \mathfrak{W}_-^{[\perp]} = h_1$, as claimed.

Step 3: *Proof of* (6.7) *with* $n = 0$. Formula (6.12) also holds with w^k replaced by $S^{-1}w^k$, and by applying $\Gamma_{\mathfrak{W}}$ to $\pi_- S^{-1}w^k + \mathfrak{W}_-^{[\perp]}$ we get

$$\pi_+ S^{-1}w^k + \mathfrak{W}_+ = \Gamma_{\mathfrak{W}}\big(\pi_- S^{-1}w^k + \mathfrak{W}_-^{[\perp]}\big) \to \Gamma_{\mathfrak{W}}\big(\pi_- S^{-1}w + \mathfrak{W}_-^{[\perp]}\big) \quad \text{in } \mathcal{H}(\mathfrak{W}_+)$$

as $k \to \infty$. By Theorem 4.1, the restriction of the quotient map $w(\cdot) \to w(\cdot) + \mathfrak{W}_+$ to $\mathcal{K}(\mathfrak{W}_+)$ is a closed operator $k^2_+(\mathcal{W}) \to \mathcal{H}(\mathfrak{W}_+)$, and, recalling also (6.14), we get (6.7) with $n = 0$.

Step 4: *Proof of* (6.7) *and* (6.8) *by induction*. Suppose that (6.7) and (6.8) hold with n replaced by $m \geqslant 0$. Then (6.6) holds with w_- replaced by $\tilde{w}_- := \pi_- S^{-m} w$ and w_+ replaced by $\tilde{w}_+ := \pi_+ S^{-m} w$. We can then repeat steps 2 and 3 above with w_- replaced by $\tilde{w}_-$ and w_+ replaced by $\tilde{w}_+$ to get (6.7) and (6.8) with n replaced by $m + 1$.

Step 5: *Proof* (6.9) *and* (6.10). The assumption of Lemma 6.8 is still satisfied if we replace w by $w^k - w$ (see, in particular, (6.12)), and hence (6.8) holds if we replace w by $w^k - w$. If we furthermore replace n by $\ell = 0, 1, \dots, n$ and add the resulting identities, then we get

$$\begin{aligned} &\left\| \pi_- S^{-n-1}\big(w^k - w\big) + \mathfrak{W}_-^{[\perp]} \right\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} \\ &\quad = \left\| \pi_-\big(w^k - w\big) + \mathfrak{W}_-^{[\perp]} \right\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} \\ &\qquad + \sum_{\ell=0}^{n} \big[w^\ell(n) - w_+(n), w^\ell(n) - w_+(n)\big]_{\mathcal{W}}. \end{aligned} \tag{6.16}$$

Here the right-hand side tends to zero as $k \to \infty$, uniformly in $n \in \mathbb{Z}^+$, and consequently $\pi_- S^{-n} w^k + \mathfrak{W}_-^{[\perp]} \to \pi_- S^{-n} w + \mathfrak{W}_-^{[\perp]}$ in $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ as $k \to \infty$, uniformly in $n \in \mathbb{Z}^+$. The uniform convergence of $\pi_+ S^{-n} w^k + \mathfrak{W}_+$ to $\pi_+ S^{-n} w_+ + \mathfrak{W}_+$ in $\mathcal{H}(\mathfrak{W}_+)$ then follows from (6.7) with w replaced by $w^k - w$. $\square$

Lemma 6.9. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with input map $\mathfrak{B}_\Sigma$, past behavior $\mathfrak{W}_{\text{past}}$, future behavior $\mathfrak{W}_{\text{fut}}$, and past/future map Γ_Σ. Then the following two conditions are equivalent*:

(1) $(x(\cdot), w_+(\cdot))$ *is a stable future trajectory of Σ satisfying $x(0) \in \mathcal{R}(\mathfrak{B}_\Sigma)$;*
(2) *There exists some $w_-(\cdot) \in \mathcal{K}(\mathfrak{W}_{\text{past}}^{[\perp]})$ such that*

$$\begin{aligned} w_+ &\in \mathcal{K}(\mathfrak{W}_{\text{fut}}), \\ w_+ + \mathfrak{W}_{\text{fut}} &= \Gamma_{\mathfrak{W}}\big(w_- + \mathfrak{W}_{\text{past}}^{[\perp]}\big), \\ x(n) &= \mathfrak{B}_\Sigma\big(\pi_- S^{-n}(w_- + w_+) + \mathfrak{W}_{\text{past}}^{[\perp]}\big), \quad n \in \mathbb{Z}^+ \end{aligned} \tag{6.17}$$

(*in particular, $x(0) = \mathfrak{B}_\Sigma(w_- + \mathfrak{W}_{\text{past}}^{[\perp]})$*).

When these equivalent conditions hold, then (6.17) *remains true for every $w_-(\cdot) \in \mathcal{K}(\mathfrak{W}_{\text{past}}^{[\perp]})$ satisfying $x(0) = \mathfrak{B}_\Sigma(w_- + \mathfrak{W}_{\text{past}}^{[\perp]})$.*

Proof. We first suppose that $(x(\cdot), w(\cdot))$ is a stable future trajectory of Σ satisfying $x(0) \in \mathcal{R}(\mathfrak{B}_\Sigma)$ and show that (6.17) holds for every $w_-(\cdot) \in \mathcal{K}(\mathfrak{W}_{\text{past}}^{[\perp]})$ satisfying $x(0) = \mathfrak{B}_\Sigma(w_- + \mathfrak{W}_{\text{past}}^{[\perp]})$.

That $w_+ \in \mathcal{K}(\mathfrak{W}_{\text{fut}})$ follows from Lemma 5.1. By assumption, $\begin{bmatrix} x(1) \\ x(0) \\ w_+(0) \end{bmatrix} \in V$ and $x(0) = \mathfrak{B}_\Sigma(w_- + \mathfrak{W}_{\text{past}}^{[\perp]})$ for some $w_- \in \mathcal{K}(\mathfrak{W}_{\text{past}}^{[\perp]})$. By Lemma 5.2, $\mathfrak{C}_\Sigma x(0) = w_+ + \mathfrak{W}_{\text{fut}}$, and hence

$w_+ + \mathfrak{W}_{\text{fut}} = \mathfrak{C}_\Sigma \mathfrak{B}_\Sigma (w_- + \mathfrak{W}_{\text{past}}^{[\perp]}) = \Gamma_\Sigma (w_- + \mathfrak{W}_{\text{past}}^{[\perp]})$. This proves the first two claims in (6.17), and it remains to prove the formula for $x(n)$ given in (6.17) for $n \geqslant 1$.

Denote $w = w_- + w_+$. By Lemma 6.8, there exists a sequence $w^k \in \mathfrak{W}_{\text{full}}$ such that $\pi_+ w^k \to w_+$ in $k_+^2(\mathcal{W})$ as $k \to \infty$ and $\pi_- S^{-n} w^k + \mathfrak{W}_{\text{past}}^{[\perp]} \to \pi_- S^{-n} w + \mathfrak{W}_{\text{past}}^{[\perp]}$ in $\mathcal{H}(\mathfrak{W}_{\text{past}}^{[\perp]})$ as $k \to \infty$, uniform in $n \in \mathbb{Z}^+$. Let $(x^k(\cdot), w^k(\cdot))$ be the externally generated stable full trajectory of Σ whose signal part is $w^k(\cdot)$ (cf. Lemma 2.5). By Lemma 5.12, $x^k(n) = \mathfrak{B}_\Sigma(\pi_- S^{-n} w^k + \mathfrak{W}_{\text{past}}^{[\perp]})$, which tends to $x_1(n) := \mathfrak{B}_\Sigma(\pi_- S^{-n} w + \mathfrak{W}_{\text{past}}^{[\perp]})$ as $k \to \infty$, uniformly in $n \in \mathbb{Z}^+$. In particular, $x^k(0) \to \mathfrak{B}_\Sigma(w_- + \mathfrak{W}_{\text{past}}^{[\perp]}) = x_1(0) = x(0)$. Since the restriction of $(x^k(\cdot), w^k(\cdot))$ to $\mathbb{Z}^+$ is a future trajectory of Σ for each k, it follows from part (1) of Lemma 2.3 that the limit $(x_1(\cdot), w_+(\cdot))$ is a stable future trajectory of Σ. This trajectory has both the same initial state $x(0)$ and the same signal part $w_+(\cdot)$ as the given trajectory $(x(\cdot), w_+(\cdot))$, and hence $x_1(n) = x(n)$ for all $n \in \mathbb{Z}^+$. This proves that the last claim in (6.17) holds.

The proof of the converse direction is based on induction over the length of the interval where $(x(\cdot), w(\cdot))$ is a solution of Σ. We begin by showing that if (6.17) holds, then $(x(\cdot), w(\cdot))$ is a trajectory of Σ on the one-point interval $[0, 0] = \{0\}$.

Suppose that (6.17) holds for $n = 0, 1$. Thus, in particular, $x(0) = \mathfrak{B}_\Sigma(w_- + \mathfrak{W}_{\text{past}}^{[\perp]})$ and $w_+ + \mathfrak{W}_{\text{fut}} = \Gamma_\Sigma(w_- + \mathfrak{W}_{\text{past}}^{[\perp]})$. By Lemma 6.3, $w_+ + \mathfrak{W}_{\text{fut}} = \mathfrak{C}_\Sigma x_0$. By part (3) of Lemma 5.7, $\left[\begin{smallmatrix} x(1) \\ x(0) \\ w(0) \end{smallmatrix}\right] \in V$ for some $x(1) \in \mathcal{X}$. By part (7) of Lemma 2.3, there exists a stable future trajectory $(x_1(\cdot), w_1(\cdot))$ of Σ satisfying $x_1(0) = x(0)$ and $w_1(0) = w_+(0)$. By the first part of the proof, $x_1(1) = \mathfrak{B}_\Sigma(\pi_- S_-^{-1}(w_- + w_1) + \mathfrak{W}_{\text{past}}^{[\perp]})$. Here $\pi_- S_-^{-1}(w_- + w_1) = \pi_- S_-^{-1}(w_- + w_+)$ since $w_1(0) = w_+(0)$, and hence $x_1(1) = \mathfrak{B}_\Sigma(\pi_- S_-^{-1}(w_- + w_+) + \mathfrak{W}_{\text{past}}^{[\perp]})$. Since we assume that (6.17) holds (for $n = 1$), we get $x(1) = x_1(1)$, and consequently $\left[\begin{smallmatrix} x(1) \\ x(0) \\ w_+(0) \end{smallmatrix}\right] \in V$. This proves that $(x(\cdot), w_+(\cdot))$ is a trajectory of Σ on the one-point interval $\{0\}$.

One can use essentially the same argument to show that if we know that $(x(\cdot), w_+(\cdot))$ is a trajectory of Σ on an interval $[0, k]$, then it is also a trajectory on $[0, k+1]$, i.e., one shifts the trajectory $k+1$ steps to the left, and then apply the above argument. The invariance of the first two conditions in (6.17) under this left-shift follows from Lemma 6.8. Thus, by induction, $(x(\cdot), w(\cdot))$ is a future trajectory of Σ. By Lemma 2.1, this trajectory is stable. □

7. The observable backward conservative realization

In this section we shall construct a canonical model $\Sigma_{\text{obc}}^{\mathfrak{W}_+} = (V_{\text{obc}}^{\mathfrak{W}_+}; \mathcal{X}_{\text{obc}}^{\mathfrak{W}_+}, \mathcal{W})$ of a passive observable backward conservative s/s system with a given passive future behavior $\mathfrak{W}_+$.

Theorem 7.1. *Let $\mathfrak{W}_+$ be a passive future behavior on the Kreĭn space $\mathcal{W}$. Let $\mathcal{X}_{\text{obc}}^{\mathfrak{W}_+} = \mathcal{H}(\mathfrak{W}_+)$, where $\mathcal{H}(\mathfrak{W}_+)$ is the space defined in Theorem* 4.4, *and let*

$$V_{\text{obc}}^{\mathfrak{W}_+} = \left\{ \begin{bmatrix} S_+^* w + \mathfrak{W}_+ \\ w + \mathfrak{W}_+ \\ w(0) \end{bmatrix} \in \begin{bmatrix} \mathcal{H}(\mathfrak{W}_+) \\ \mathcal{H}(\mathfrak{W}_+) \\ \mathcal{W} \end{bmatrix} \;\middle|\; w \in \mathcal{K}(\mathfrak{W}_+) \right\}, \tag{7.1}$$

where $\mathcal{K}(\mathfrak{W}_+)$ is the space defined in (4.20). *Then $\Sigma_{\text{obc}}^{\mathfrak{W}_+} = (V_{\text{obc}}^{\mathfrak{W}_+}; \mathcal{H}(\mathfrak{W}_+), \mathcal{W})$ is a passive observable backward conservative s/s system whose future behavior is equal to $\mathfrak{W}_+$. Moreover, $(x(\cdot), w(\cdot))$ is a stable future trajectory of $\Sigma_{\text{obc}}^{\mathfrak{W}_+}$ if and only if*

$$w \in \mathcal{K}(\mathfrak{W}_+) \quad \textit{and} \quad x(n) = \left(S_+^*\right)^n w + \mathfrak{W}_+, \quad n \in \mathbb{Z}^+. \tag{7.2}$$

Proof. In this proof we denote the node space of $\Sigma_{\text{obc}}^{\mathfrak{W}_+}$ by $\mathfrak{K}_+ := -\mathcal{H}(\mathfrak{W}_+)\,[\dot{+}]\,\mathcal{H}(\mathfrak{W}_+)\,[\dot{+}]\,\mathcal{W}$.

Step 1: *$V_{\text{obc}}^{\mathfrak{W}_+}$ is a nonnegative subspace of* $\mathfrak{K}_+$. It follows from Lemma 4.3 that $V_{\text{obc}}^{\mathfrak{W}_+} \subset \mathfrak{K}_+$, and that $V_{\text{obc}}^{\mathfrak{W}_+}$ is nonnegative in $\mathfrak{K}_+$. It is a subspace of $\mathfrak{K}_+$ since it is a linear image of the subspace $\mathcal{K}(\mathfrak{W}_+)$ of $k_+^2(\mathcal{W})$.

Step 2: *$V_{\text{obc}}^{\mathfrak{W}_+}$ is closed and $(V_{\text{obc}}^{\mathfrak{W}_+})^{[\perp]} \subset V_{\text{obc}}^{\mathfrak{W}_+}$*. Define $\mathring{V}_{\text{obc}}$ by

$$\mathring{V}_{\text{obc}} = \left\{ \begin{bmatrix} S_+^* z^\dagger + \mathfrak{W}_+ \\ z^\dagger + \mathfrak{W}_+ \\ z^\dagger(0) \end{bmatrix} \,\middle|\, z^\dagger \in \mathfrak{W}_+^{[\perp]} \right\}. \tag{7.3}$$

Then $\mathring{V}_{\text{obc}} \subset V_{\text{obc}}^{\mathfrak{W}_+}$ since $\mathcal{H}^0(\mathfrak{W}_+) \subset \mathcal{H}(\mathfrak{W}_+)$. We claim that $(\mathring{V}_{\text{obc}})^{[\perp]} = V_{\text{obc}}^{\mathfrak{W}_+}$. Clearly, this implies that $V_{\text{obc}}^{\mathfrak{W}_+}$ is closed, and that $(V_{\text{obc}}^{\mathfrak{W}_+})^{[\perp]} \subset V_{\text{obc}}^{\mathfrak{W}_+}$ since $(V_{\text{obc}}^{\mathfrak{W}_+})^{[\perp]} = ((\mathring{V}_{\text{obc}})^{[\perp]})^{[\perp]}$ is the closure of $\mathring{V}_{\text{obc}}$.

A vector $k = \left[\begin{smallmatrix} x_1 \\ x_0 \\ w_0 \end{smallmatrix}\right]$ belongs to $(\mathring{V}_{\text{obc}})^{[\perp]}$ if and only if $x_1, x_0 \in \mathcal{K}(\mathfrak{W}_+)$, $w_0 \in \mathcal{W}$, and

$$-\left(x_1, S_+^* z^\dagger + \mathfrak{W}_+\right)_{\mathcal{H}(\mathfrak{W}_+)} + \left(x_0, z^\dagger + \mathfrak{W}_+\right)_{\mathcal{H}(\mathfrak{W}_+)} + \left[w_0, z^\dagger(0)\right]_{\mathcal{W}} = 0, \quad z^\dagger \in \mathfrak{W}_+^{[\perp]}. \tag{7.4}$$

Since $\mathfrak{W}_+$ is S_+-invariant, its orthogonal companion $\mathfrak{W}_+^{[\perp]}$ is S_+^*-invariant, i.e., $S_+^* z^\dagger \in \mathfrak{W}_+^{[\perp]}$ whenever $z^\dagger \in \mathfrak{W}_+^{[\perp]}$. By (4.14), for every $v_1 \in x_1$ and $v_0 \in x_0$, (7.4) can therefore be rewritten in the form

$$\left[v_1, S_+^* z^\dagger\right]_{k_+^2(\mathcal{W})} - \left[v_0, z^\dagger\right]_{k_+^2(\mathcal{W})} + \left[w_0, z^\dagger(0)\right]_{\mathcal{W}} = 0, \quad z^\dagger \in \mathfrak{W}_+^{[\perp]}. \tag{7.5}$$

Define the sequence $w \in k_+^2(\mathcal{W})$ by $w(0) = w_0$ and $w(n) = 0$ for $n > 0$, and let P_0 be the orthogonal projection in $k_+^2(\mathcal{W})$ onto the subspace of vectors $k(\cdot)$ satisfying $k(n) = 0$ for $n > 0$. Then (7.5) can be rewritten as

$$\left[S_+ v_1 - v_0 + P_0 w, z^\dagger\right]_{k_+^2(\mathcal{W})} = 0, \quad z^\dagger \in \mathfrak{W}_+^{[\perp]}.$$

Since $(\mathfrak{W}_+^{[\perp]})^{[\perp]} = \mathfrak{W}_+$, this is equivalent to

$$S_+ v_1 - v_0 + P_0 w = z$$

for some $z \in \mathfrak{W}_+$. Define $v = v_0 + z$. Then $v \in x_0$, and

$$S_+ v_1 - v + P_0 w = 0.$$

This is equivalent to the pair of equations

$$v(0) = w_0 \quad \text{and} \quad v_1 = S_+^* v.$$

Thus, $\begin{bmatrix} x_1 \\ x_0 \\ w_0 \end{bmatrix} \in (\mathring{V}_{\text{obc}})^{[\perp]}$ if and only if $x_0 = v + \mathfrak{W}_+$, $x_1 = S_+^* v + \mathfrak{W}_+$, and $w_0 = v(0)$ for some $v \in \mathcal{K}(\mathfrak{W}_+)$, or equivalently, if and only if $k \in V_{\text{obc}}^{\mathfrak{W}_+}$.

Step 3: $V_{\text{obc}}^{\mathfrak{W}_+}$ *is the generating subspace of a passive and backward conservative s/s system* $\Sigma_{\text{obc}}^{\mathfrak{W}_+} = (V_{\text{obc}}^{\mathfrak{W}_+}; \mathcal{H}(\mathfrak{W}_+), \mathcal{W})$. By steps 1 and 2, $V_{\text{obc}}^{\mathfrak{W}_+}$ is closed and nonnegative, and $(V_{\text{obc}}^{\mathfrak{W}_+})^{[\perp]}$ is neutral, hence nonpositive. By, e.g., [3, Proposition 2.2(5)], $V_{\text{obc}}^{\mathfrak{W}_+}$ is a maximal nonnegative subspace of $\mathfrak{K}_+$, and hence, by [3, Corollary 5.13], it generates a passive backward conservative s/s system.

Step 4: $\Sigma_{\text{obc}}^{\mathfrak{W}_+}$ *is observable*. Let $(x(\cdot), w(\cdot))$ be an unobservable future trajectory of Σ, i.e., $w(n) = 0$ for all $n \in \mathbb{Z}^+$. Let $z^\dagger \in \mathfrak{W}_+^{[\perp]}$, and define $x^\dagger(n) = (S_+^*)^n z^\dagger + \mathfrak{W}_+$, $n \in \mathbb{Z}^+$. Then it follows from (7.3) that $\begin{bmatrix} x^\dagger(n+1) \\ x^\dagger(n) \\ z^\dagger(n) \end{bmatrix} \in \mathring{V}_{\text{obc}}$ for all $n \in \mathbb{Z}^+$. Since $\mathring{V}_{\text{obc}} \subset V_{\text{obc}}^{[\perp]}$, this means that $(x^\dagger(\cdot), z^\dagger(\cdot))$ is a future trajectory of the anti-passive dual of $\Sigma_{\text{obc}}^{\mathfrak{W}_+}$ (and also a future trajectory of $\Sigma_{\text{obc}}^{\mathfrak{W}_+}$). Moreover, $x^\dagger(n) \to 0$ in $\mathcal{H}(\mathfrak{W}_+)$ as $n \to \infty$, because by Theorem 4.1,

$$\|x^\dagger(n)\|^2_{\mathcal{H}(\mathfrak{W}_+)} = -\big[(S_+^*)^n z^\dagger, (S_+^*)^n z^\dagger\big]_{k_+^2(\mathcal{W})} = -\big[z^\dagger, z^\dagger\big]_{k^2([n,\infty);\mathcal{W})}$$

which tends to zero as $k \to \infty$. By part (3) of Lemma 3.2 and Theorem 4.1 (recall that $w(\cdot) = 0$),

$$\big(x(0), z^\dagger + \mathfrak{W}_+\big)_{\mathcal{H}(\mathfrak{W}_+)} = \big(x(0), x^\dagger(0)\big)_{\mathcal{H}(\mathfrak{W}_+)} = -\big[w(\cdot), z^\dagger(\cdot)\big]_{k_+^2(\mathcal{W})} = 0.$$

Thus, $x(0)$ is orthogonal to $\mathcal{H}^0(\mathfrak{W}_+)$, and since $\mathcal{H}^0(\mathfrak{W}_+)$ is dense in $\mathcal{H}(\mathfrak{W}_+)$, this implies that $x(0) = 0$. Thus, $\Sigma_{\text{obc}}^{\mathfrak{W}_+}$ is observable.

Step 5: *If* (7.2) *holds, then* $(x(\cdot), w(\cdot))$ *is a stable future trajectory of* $\Sigma_{\text{obc}}^{\mathfrak{W}_+}$. Let $w \in \mathcal{K}(\mathfrak{W}_+)$ and define $x(n) = (S_+^*)^n w + \mathfrak{W}_+$, $n \geqslant 0$. Then it is easy to see that $(x(\cdot), w(\cdot))$ is a trajectory of $\Sigma_{\text{obc}}^{\mathfrak{W}_+}$ with $w \in k_+^2(\mathcal{W})$. It is stable since $\Sigma_{\text{obc}}^{\mathfrak{W}_+}$ is passive and $w(\cdot) \in \ell_+^2(\mathcal{W})$ (see Lemma 2.1).

Step 6: *The future behavior of* $\Sigma_{\text{obc}}^{\mathfrak{W}_+}$ *is equal to* $\mathfrak{W}_+$. It follows from step 5 that the future behavior $\mathfrak{W}_+^\Sigma$ of $\Sigma_{\text{obc}}^{\mathfrak{W}_+}$ contains $\mathfrak{W}_+$, and hence $\mathfrak{W}_+^\Sigma = \mathfrak{W}_+$ since, by Theorem 2.8, $\mathfrak{W}_+^\Sigma$ is nonnegative, and by assumption, $\mathfrak{W}_+$ is maximal nonnegative in $k_+^2(\mathcal{W})$.

Step 7: *If* $(x(\cdot), w(\cdot))$ *is a stable future trajectory of* $\Sigma_{\text{obc}}^{\mathfrak{W}_+}$, *then* (7.2) *holds*. Let $(x(\cdot), w(\cdot))$ be a stable future trajectory of $\Sigma_{\text{obc}}^{\mathfrak{W}_+}$. By Lemma 5.1, $w(\cdot) \in \mathcal{K}(\mathfrak{W}_+)$. As we saw above, if we define $x_1(n) = (S_+^*)^n w + \mathfrak{W}_+$, $n \in \mathbb{Z}^+$, then $(x_1(\cdot), w(\cdot))$ is another stable future trajectory of $\Sigma_{\text{obc}}^{\mathfrak{W}_+}$ with the same signal part $(\cdot)$. Since $\Sigma_{\text{obc}}^{\mathfrak{W}_+}$ is observable, this implies that $x(n) = x_1(n)$ for all $n \in \mathbb{Z}^+$, i.e., (7.2) holds. □

Definition 7.2. We call the system $\Sigma_{\text{obc}}^{\mathfrak{W}_+}$ the *canonical model* of an observable passive backward conservative s/s system with future behavior $\mathfrak{W}_+$.

Corollary 7.3. *The system $\Sigma_{\mathrm{obc}}^{\mathfrak{W}_+}$ is approximately null-controllable, i.e., the set of all the initial states $x(0)$ of all those future trajectories of $\Sigma_{\mathrm{obc}}^{\mathfrak{W}_+}$ with have finite support is dense in $\mathcal{X}_{\mathrm{obc}} = \mathcal{H}(\mathfrak{W}_+)$.*

Proof. This follows from Theorem 7.1 and Lemma 4.2. □

In Theorem 2.11 we established the connections (2.15)–(2.17) between passive past, future, and full behaviors $\mathfrak{W}_- = \mathfrak{W}_-$, $\mathfrak{W}_+ = \mathfrak{W}_+$, and $\mathfrak{W} = \mathfrak{W}$ (see Remark 2.13). In particular, they permit us to define unique full behavior $\mathfrak{W}$ in terms of a given future behavior $\mathfrak{W}_+$. Once we have the full behavior $\mathfrak{W}$, we can also define the past/future map $\Gamma_{\mathfrak{W}}$ by (6.1).

Lemma 7.4. *The input map of $\Sigma_{\mathrm{obc}}^{\mathfrak{W}_+}$ is the past/future map $\Gamma_{\mathfrak{W}}$ of $\mathfrak{W}$, and the output map of $\Sigma_{\mathrm{obc}}^{\mathfrak{W}_+}$ is the identity on $\mathcal{H}(\mathfrak{W}_+)$.*

Proof. It follows from Lemma 5.4 and Theorem 7.1 that we for every stable future trajectory $(x(\cdot), w(\cdot))$ of $\Sigma_{\mathrm{obc}}^{\mathfrak{W}_+}$ have

$$\mathfrak{C}_{\Sigma_{\mathrm{obc}}^{\mathfrak{W}_+}} x(0) = w + \mathfrak{W}_+ = x(0).$$

Thus, the output map of $\Sigma_{\mathrm{obc}}^{\mathfrak{W}_+}$ is the identity. This implies that the input map of $\Sigma_{\mathrm{obc}}^{\mathfrak{W}_+}$ is $\Gamma_{\mathfrak{W}}$, since the product of the input and output maps must be equal to $\Gamma_{\mathfrak{W}}$. □

Lemma 7.5. *A sequence $(x(\cdot), w(\cdot))$ is an externally generated stable past trajectory of Σ_{obc} if and only if $w \in \mathfrak{W}_-$ and $x(-n) = \Gamma_{\mathfrak{W}}(S_-^n w + \mathfrak{W}_-^{[\perp]})$, $n \geqslant 0$.*

Proof. This follows from Lemmas 5.4 and 7.4. □

Definition 7.6. A bounded linear operator $E : \mathcal{X}_1 \to \mathcal{X}_2$ *intertwines* the two passive s/s systems $\Sigma_1 = (V_1; \mathcal{X}_1; \mathcal{W})$ and $\Sigma_2 = (V_2; \mathcal{X}_2; \mathcal{W})$ (with the same signal space $\mathcal{W}$) if

$$\begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V_1 = V_2 \cap \begin{bmatrix} \mathcal{X}_2 \\ \mathcal{R}(E) \\ \mathcal{W} \end{bmatrix}. \tag{7.6}$$

In this case we say that Σ_1 and Σ_2 are *boundedly intertwined by E*, or *contractively intertwined by E* if E is a contraction. If E has a bounded inverse, then we say that Σ_1 and Σ_2 are *similar* with similarity operator E, and if E is unitary, then we say that Σ_1 and Σ_2 are *unitarily similar*.

It is also possible to define a more general intertwinement relation where E is allowed to be a closed relation instead of a bounded operator, but Definition 7.6 covers our present needs.

Lemma 7.7. *The two passive s/s systems $\Sigma_1 = (V_1; \mathcal{X}_1; \mathcal{W})$ and $\Sigma_2 = (V_2; \mathcal{X}_2; \mathcal{W})$ are intertwined by the operator $E \in \mathcal{B}(\mathcal{X}_1; \mathcal{X}_2)$ if and only if the formula*

$$\big(x_1(\cdot), w(\cdot)\big) \mapsto \big(Ex_1(\cdot), w(\cdot)\big) \tag{7.7}$$

defines a map from the set of all stable future trajectories $(x_1(\cdot), w(\cdot))$ *of* Σ_1 *onto the set of all stable future trajectories* $(x_2(\cdot), w(\cdot))$ *of* Σ_2 *satisfying* $x_2(0) \in \mathcal{R}(E)$*. In particular, if* Σ_1 *and* Σ_2 *are boundedly intertwined by E, then they have the same future behavior.*

Proof. Let us first comment on the last claim: For externally generated trajectories of Σ_2 the condition $x_2(0) \in \mathcal{R}(E)$ is trivially true, and so there is a one-to-one correspondence between the externally generated future trajectories of Σ_1 and Σ_2 (an externally generated trajectory is uniquely determined by its signal part $w(\cdot)$). This implies that the two systems have the same future behavior.

Suppose next that (7.6) holds, i.e., that E intertwines Σ_1 and Σ_2. Then trivially, if $(x_1(\cdot), w(\cdot))$ is a stable future trajectory of Σ_1, then $(Ex_1(\cdot), w(\cdot))$ is a stable future trajectory of Σ_2. Conversely, suppose that $(x_2(\cdot), w(\cdot))$ is a stable future trajectory of Σ_2. Then

$$\begin{bmatrix} x_2(n+1) \\ x_2(n) \\ w(n) \end{bmatrix} \in V_2, \quad n \in \mathbb{Z}^+. \tag{7.8}$$

Taking $n = 1$ above we can use (7.6) to conclude that there exists a vector $\left[\begin{smallmatrix} x_1(1) \\ x_1(0) \\ w(0) \end{smallmatrix}\right] \in V_1$ such that $\left[\begin{smallmatrix} x_2(1) \\ x_2(0) \\ w(0) \end{smallmatrix}\right] = \left[\begin{smallmatrix} Ex_1(1) \\ Ex_1(0) \\ w(0) \end{smallmatrix}\right]$. In particular, $x_2(1) \in \mathcal{R}(E)$. We can therefore repeat the same argument with $n = 1$ to conclude that there exists (a unique) $x_1(2) \in \mathcal{X}_1$ such that $\left[\begin{smallmatrix} x_1(2) \\ x_1(1) \\ w(1) \end{smallmatrix}\right] \in V_1$ and $x_2(2) = Ex_1(2)$. By repeating this argument indefinitely (or by using induction) we get a sequence $x_1(\cdot)$ such that $(x_1(\cdot), w(\cdot))$ is a future trajectory of Σ_1, and such that $x_2(\cdot) = Ex_1(\cdot)$. By Lemma 2.1, the trajectory $(x_1(\cdot), w(\cdot))$ is stable. Thus, the mapping defined in (7.7) is surjective.

We then turn to the converse statement, and suppose that the stable future trajectories of Σ_1 and Σ_2 are related as described in the lemma. Let $(x_1(\cdot), w(\cdot))$ be a stable future trajectory of Σ_1. Then, by the assumption, $(Ex_1(\cdot), w(\cdot))$ be a stable future trajectory of Σ_1. In particular, $\left[\begin{smallmatrix} Ex_1(1) \\ Ex_1(0) \\ w(0) \end{smallmatrix}\right] \in V_2$. By part (7) of Lemma 2.3, the vector $\left[\begin{smallmatrix} x(1) \\ x(0) \\ w(0) \end{smallmatrix}\right]$ can be an arbitrary vector in V. This shows that the that the left-hand side of (7.6) is a subset of the right-hand side. On the other hand, if $(x_2(\cdot), w(\cdot))$ is an arbitrary stable future trajectory of Σ_2 satisfying $x_2(0) \in \mathcal{R}(E)$, then by assumption, there exists a future trajectory $(x_1(\cdot), w_1(\cdot))$ of Σ_1 such that $x_2(\cdot) = Ex_1(\cdot)$. Here $\left[\begin{smallmatrix} x_2(1) \\ x_2(0) \\ w \end{smallmatrix}\right]$ represents an arbitrary vector in the right-hand side of (7.6), and we have shown that it belongs to the left-hand side of (7.6). Thus, we have equality in (7.6). □

Theorem 7.8. *Let* $\Sigma = (V; \mathcal{X}, \mathcal{W})$ *be a passive s/s system with output map* $\mathfrak{C}_\Sigma$ *and future behavior* $\mathfrak{W}_+$*. Then* Σ *and* $\Sigma_{\text{obc}}^{\mathfrak{W}_+} = (V_{\text{obc}}^{\mathfrak{W}_+}; \mathcal{X}_{\text{obc}}^{\mathfrak{W}_+}, \mathcal{W})$ *are contractively intertwined by* $\mathfrak{C}_\Sigma$*, i.e.,*

$$\begin{bmatrix} \mathfrak{C}_\Sigma & 0 & 0 \\ 0 & \mathfrak{C}_\Sigma & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V = V_{\text{obc}}^{\mathfrak{W}_+} \cap \begin{bmatrix} \mathcal{X}_{\text{obc}}^{\mathfrak{W}_+} \\ \mathfrak{S}_{\text{fut}}^{\Sigma} \\ \mathcal{W} \end{bmatrix}. \tag{7.9}$$

Proof. Let $(x(\cdot), w(\cdot))$ be a stable future trajectory of Σ. By Lemmas 5.1 and 5.4, $w(\cdot) \in \mathcal{K}(\mathfrak{W}_{\text{fut}})$ and $\mathfrak{C}_\Sigma x(n) = (S_+^*)^n w + \mathfrak{W}_{\text{fut}}$, $n \in \mathbb{Z}^+$. Define $x_0(\cdot) = \mathfrak{C}_\Sigma x(\cdot)$. Then $x_0(n) = (S_+^*)^n w + \mathfrak{W}_{\text{fut}}$,

$n \in \mathbb{Z}^+$ (where $\mathfrak{W}_{\mathrm{fut}}$ is the future behavior of Σ), and by Theorem 7.1, $(x_0(\cdot), w(\cdot))$ is stable future trajectory of $\Sigma_{\mathrm{obc}}^{\mathfrak{W}_+}$. By part (7) of Lemma 2.3, the vector $\left[\begin{smallmatrix} x(1) \\ x(0) \\ w(0) \end{smallmatrix}\right]$ can be an arbitrary vector in V, and since $\left[\begin{smallmatrix} x_0(1) \\ x_0(0) \\ w(0) \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathfrak{C}_\Sigma x(1) \\ \mathfrak{C}_\Sigma x(0) \\ w(0) \end{smallmatrix}\right] \in V_{\mathrm{obc}}^{\mathfrak{W}_+}$, this implies that the left-hand side of (7.9) is a subset of the right-hand side.

To prove the converse inclusion we let $(x_0(\cdot), w(\cdot))$ be a stable future trajectory of $\Sigma_{\mathrm{obc}}^{\mathfrak{W}_+}$, and suppose that $x_0(0) \in \mathfrak{S}_{\mathrm{fut}}^{\Sigma}$. Then, by part (7) of Lemma 2.3, the vector $\left[\begin{smallmatrix} x_0(1) \\ x_0(0) \\ w(0) \end{smallmatrix}\right]$ represents an arbitrary vector in the right-hand side of (7.9). Choose some arbitrary $x(0) \in \mathcal{X}$ such that $\mathfrak{C}_\Sigma x(0) = x_0(0)$. Recall that the output map of $\Sigma_{\mathrm{obc}}^{\mathfrak{W}_+}$ is the identity. By part (3) of Lemma 5.7 applied to $\Sigma_{\mathrm{obc}}^{\mathfrak{W}_+}$, $w_0 \in (\mathfrak{C}_\Sigma x_0(0))(0)$, and by the same lemma applied to the system Σ, there exists some $x(1) \in \mathcal{X}$ such that $\left[\begin{smallmatrix} x(1) \\ x(0) \\ w(0) \end{smallmatrix}\right] \in V$. By the first inclusion that we already proved, this implies that $\left[\begin{smallmatrix} \mathfrak{C}_\Sigma x(1) \\ \mathfrak{C}_\Sigma x(0) \\ w(0) \end{smallmatrix}\right] \in V_{\mathrm{obc}}^{\mathfrak{W}_+}$. But here the last two components of $V_{\mathrm{obc}}^{\mathfrak{W}_+}$ determine the first component uniquely, and hence we must have $x_0(0) = \mathfrak{C}_\Sigma x(0)$. Thus, $\left[\begin{smallmatrix} x_0(1) \\ x_0(0) \\ w(0) \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathfrak{C}_\Sigma x(1) \\ \mathfrak{C}_\Sigma x(0) \\ w(0) \end{smallmatrix}\right]$, where $\left[\begin{smallmatrix} x(1) \\ x(0) \\ w(0) \end{smallmatrix}\right] \in V$. This proves that the right-hand side of (7.8) is contained in the left-hand side. □

Corollary 7.9. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with output map $\mathfrak{C}_\Sigma$ and full behavior $\mathfrak{W}$, and let $\Sigma_{\mathrm{obc}}^{\mathfrak{W}_+}$ be the canonical model of an observable backward conservative s/s system with full behavior $\mathfrak{W}$. Then the formula*

$$\big(x(\cdot), w(\cdot)\big) \mapsto \big(\mathfrak{C}_\Sigma x(\cdot), w(\cdot)\big) \tag{7.10}$$

defines a map from the set of all stable future trajectories of Σ onto the set of all stable future trajectories $(x_0(\cdot), w(\cdot))$ of $\Sigma_{\mathrm{obc}}^{\mathfrak{W}_+}$ satisfying $x_0(0) \in \mathcal{R}(\mathfrak{C}_\Sigma)$.

Proof. This follows from Lemma 7.7 and Theorem 7.8. □

Corollary 7.10. *Any two observable and backward conservative realizations of a given full behavior $\mathfrak{W}$ are unitarily similar to each other.*

Proof. This is true, because, by Lemma 5.20, the output maps of these two systems are unitary, and hence, by Corollary 7.9, both systems are unitarily similar to $\Sigma_{\mathrm{obc}}^{\mathfrak{W}_+}$. □

8. The controllable forward conservative realization

In this section we shall construct a canonical model $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-} = (V_{\mathrm{cfc}}^{\mathfrak{W}_-}; \mathcal{X}_{\mathrm{cfc}}^{\mathfrak{W}_-}, \mathcal{W})$ of a passive controllable forward conservative s/s system with a given passive past behavior $\mathfrak{W}_-$. The results for this model are analogous to the results on the model $\Sigma_{\mathrm{obc}}^{\mathfrak{W}_+}$ obtained in the preceding section. The state space of $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$ is the Hilbert space $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ presented in Theorem 4.4 (whereas the state space of $\Sigma_{\mathrm{obc}}^{\mathfrak{W}_+}$ is the Hilbert space $\mathcal{H}(\mathfrak{W}_+)$ presented in Theorem 4.1). The full description of the generating subspace $V_{\mathrm{cfc}}^{\mathfrak{W}_-}$ is more complicated than the description of $V_{\mathrm{obc}}^{\mathfrak{W}_+}$, and in our next theorem we first give a preliminary definition of $V_{\mathrm{cfc}}^{\mathfrak{W}_-}$ as the closure of the set

$$\mathring{V}_{\mathrm{cfc}}^{\mathfrak{W}_-} = \left\{ \begin{bmatrix} w_- + \mathfrak{W}_-^{[\perp]} \\ S_- w_- + \mathfrak{W}_-^{[\perp]} \\ w_-(-1) \end{bmatrix} \in \begin{bmatrix} \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \\ \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \\ \mathcal{W} \end{bmatrix} \;\middle|\; w_- \in \mathfrak{W}_- \right\}. \tag{8.1}$$

Since every $w_- \in \mathfrak{W}_-$ can be extended to a function $w \in \mathfrak{W}$, and since $\pi_- w \in \mathfrak{W}_-$ whenever $w \in \mathfrak{W}$, Eq. (8.1) can alternatively be written in the form (where we have shifted the extended function one step to the left)

$$\mathring{V}_{\mathrm{cfc}}^{\mathfrak{W}_-} = \left\{ \begin{bmatrix} \pi_- S^{-1} w + \mathfrak{W}_-^{[\perp]} \\ \pi_- w + \mathfrak{W}_-^{[\perp]} \\ w(0) \end{bmatrix} \in \begin{bmatrix} \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \\ \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \\ \mathcal{W} \end{bmatrix} \;\middle|\; w \in \mathfrak{W} \right\}. \tag{8.2}$$

A full description of $V_{\mathrm{cfc}}^{\mathfrak{W}_-}$ will be given later in Theorem 8.6.

Theorem 8.1. *Let $\mathcal{W}$ be a Kreĭn space, and let $\mathfrak{W}_-$ be a passive past behavior on $\mathcal{W}$. Let $\mathcal{X}_{\mathrm{cfc}}^{\mathfrak{W}_-} := \mathcal{H}(\mathfrak{W}_-^{[\perp]})$ and let $V_{\mathrm{cfc}}^{\mathfrak{W}_-}$ be the closure of the set $\mathring{V}_{\mathrm{cfc}}^{\mathfrak{W}_-}$ defined in* (8.1) *in the Kreĭn space $\mathfrak{K}_- := -\mathcal{H}(\mathfrak{W}_-^{[\perp]})\,[\dotplus]\,\mathcal{H}(\mathfrak{W}_-^{[\perp]})\,[\dotplus]\,\mathcal{W}$. Then $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-} = (V_{\mathrm{cfc}}^{\mathfrak{W}_-}; \mathcal{H}(\mathfrak{W}_-^{[\perp]}), \mathcal{W})$ is a passive controllable forward conservative s/s system whose past behavior is equal to $\mathfrak{W}_-$. Moreover, the following claims are true*:

(1) *The sequence $(x(\cdot), w(\cdot))$ is an externally generated stable past trajectory of $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$ if and only if*

$$w \in \mathfrak{W}_- \quad \textit{and} \quad x(n) = S_-^{|n|} w + \mathfrak{W}_-^{[\perp]}, \quad n \leqslant 0. \tag{8.3}$$

(2) *If $(x(\cdot), w(\cdot))$ is a stable past trajectory of $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$, then*

$$w \in \mathcal{K}\big(\mathfrak{W}_-^{[\perp]}\big) \quad \textit{and} \quad x(n) = S_-^{|n|} w + \mathfrak{W}_-^{[\perp]}, \quad n \leqslant 0. \tag{8.4}$$

Proof. *Step* 1: *$V_{\mathrm{cfc}}^{\mathfrak{W}_-}$ is a neutral subspace of $\mathfrak{K}$.* Recall that $w + \mathfrak{W}_-^{[\perp]} \in \mathcal{H}^0(\mathfrak{W}_-^{[\perp]}) \subset \mathcal{H}(\mathfrak{W}_-^{[\perp]})$ for every $w \in \mathfrak{W}_-$. Since $\mathfrak{W}_-$ is S_--invariant, it is also true that $S_- w + \mathfrak{W}_-^{[\perp]} \in \mathcal{H}(\mathfrak{W}_-^{[\perp]})$ for every $w \in \mathfrak{W}_-$. This implies that $\mathring{V}_{\mathrm{cfc}}^{\mathfrak{W}_-}$ is a subspace of $\mathfrak{K}_-$. To show that $V_{\mathrm{cfc}}^{\mathfrak{W}_-}$ is neutral it suffices to show that $\mathring{V}_{\mathrm{cfc}}^{\mathfrak{W}_-}$ is neutral, since $\mathring{V}_{\mathrm{cfc}}^{\mathfrak{W}_-}$ is dense in $V_{\mathrm{cfc}}^{\mathfrak{W}_-}$. However, this follows from Lemma 4.6.

Step 2: *$V_{\mathrm{cfc}}^{\mathfrak{W}_-}$ is maximal nonnegative in $\mathfrak{K}_-$.* Let $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ be a fundamental decomposition of $\mathcal{W}$. This induces a fundamental decomposition of the node space

$$\mathfrak{K}_- := \begin{bmatrix} -\mathcal{H}(\mathfrak{W}_-^{[\perp]}) \\ \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \\ \mathcal{W} \end{bmatrix} = \begin{bmatrix} -\mathcal{H}(\mathfrak{W}_-^{[\perp]}) \\ 0 \\ -\mathcal{Y} \end{bmatrix} [\dotplus] \begin{bmatrix} 0 \\ \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \\ \mathcal{U} \end{bmatrix}.$$

Arguing in the same way as we did in the proof of Lemma 5.16 with $x(0)$ replaced by $\pi_- w + \mathfrak{W}_-^{[\perp]}$ and $x(-1)$ replaced by $\pi_- S w + \mathfrak{W}_-^{[\perp]}$ we find that the projection of $\mathring{V}_{\mathrm{cfc}}^{\mathfrak{W}_-}$ onto the positive component of this fundamental decomposition is equal to $\left[\begin{smallmatrix} \mathcal{H}^0(\mathfrak{W}_-^{[\perp]}) \\ \mathcal{U} \end{smallmatrix}\right]$, which is dense in $\left[\begin{smallmatrix} \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \\ \mathcal{U} \end{smallmatrix}\right]$. We know that $V_{\mathrm{cfc}}^{\mathfrak{W}_-}$ is neutral, and hence it is the graph of an isometric operator $\left[\begin{smallmatrix} A_0 & B \\ C_0 & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{H}^0(\mathfrak{W}_-^{[\perp]}) \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \\ \mathcal{Y} \end{smallmatrix}\right]$ (i.e., A_0 and C_0 are defined on $\mathcal{H}^0(\mathfrak{W}_-^{[\perp]})$, and B and D are defined on $\mathcal{U}$). This implies that $\left[\begin{smallmatrix} A_0 & B_0 \\ C & D \end{smallmatrix}\right]$ has a unique extension to an isometric operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \\ \mathcal{Y} \end{smallmatrix}\right]$. Since $V_{\mathrm{cfc}}^{\mathfrak{W}_-}$ is the closure of $\mathring{V}_{\mathrm{cfc}}^{\mathfrak{W}_-}$, it is the graph of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, and hence maximal nonnegative.

Step 3: *$V_{\mathrm{cfc}}^{\mathfrak{W}_-}$ is the generating subspace of a passive and forward conservative s/s system $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-} = (V_{\mathrm{cfc}}^{\mathfrak{W}_-}; \mathcal{H}(\mathfrak{W}_-), \mathcal{W})$*. This follows from steps 1 and 2.

Step 4: *If* (8.3) *holds, then* $(x(\cdot), w(\cdot))$ *is a stable externally generated past trajectory of* $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$. When $w \in \mathfrak{W}_-$ and $x(n) = (S_-^{|n|} w) + \mathfrak{W}_-^{[\perp]}$, $n \leqslant 0$, then $\left[\begin{smallmatrix} x(n+1) \\ x(n) \\ w(n) \end{smallmatrix}\right] \in \mathring{V}_{\mathrm{cfc}}^{\mathfrak{W}_-} \subset V_{\mathrm{cfc}}^{\mathfrak{W}_-}$ for all $n \in \mathbb{Z}^-$. Thus, by definition, $(x(\cdot), w(\cdot))$ is a past trajectory of $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$. Clearly $w \in k_-^2(\mathcal{W})$. To see that $x(n) \to 0$ as $n \to -\infty$ we argue as follows. The subspace $\mathring{V}_{\mathrm{cfc}}^{\mathfrak{W}_-}$ is neutral in $\mathfrak{K}_-$, and hence, for all $n \in \mathbb{Z}^-$,

$$\|x(n)\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} = \|x(0)\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} - \sum_{k=n}^{-1} [w(k), w(k)]_{\mathcal{W}}.$$

As $n \to -\infty$, the last sum tends to $[w(\cdot), w(\cdot)]_{k_-^2(\mathcal{W})}$. However, by (4.19),

$$\|x(0)\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} = \|w + \mathcal{Z}^{[\perp]}\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} = [w(\cdot), w(\cdot)]_{k_-^2(\mathcal{W})}.$$

This implies that $x(n) \to 0$ in $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ as $n \to -\infty$.

Step 5: *The past behavior of $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$ is equal to $\mathfrak{W}_-$*. It follows from step 4 that the past behavior $\mathfrak{W}_-^{\Sigma}$ of $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$ contains $\mathfrak{W}_-$, and hence $\mathfrak{W}_-^{\Sigma} = \mathfrak{W}_-$ since, by Theorem 2.8, $\mathfrak{W}_-^{\Sigma}$ is nonnegative, and by assumption, $\mathfrak{W}_-$ is maximal nonnegative in $k_-^2(\mathcal{W})$.

Step 6: *Σ_{cfc} is controllable*. It follows from step 4 that if $w \in \mathfrak{W}_-$ has compact support, then $w + \mathfrak{W}_-^{[\perp]}$ belongs to the reachable subspace of $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$. According to Lemma 4.5, this set is dense in $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$. Thus, the set of states that can be reached in a finite time is dense in the state space $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ of $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$, and so $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$ is controllable.

Step 7: *If* $(x(\cdot), w(\cdot))$ *is a stable past trajectory of* $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$, *then* (8.4) *holds*. By Lemma 3.1, every stable past trajectory $(x(\cdot), w(\cdot))$ of $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$ is also a stable past trajectory of the anti-passive dual $\Sigma^\dagger$ of $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$. By applying the reflected version of Theorem 7.1 to the system $\Sigma^\dagger$ we find that $w \in \mathcal{H}(\mathfrak{W}_-^{[\perp]})$ and $x(-n) = (S_-^n w) + \mathfrak{W}_-^{[\perp]}$, $n \geqslant 0$.

Step 8: *If* $(x(\cdot), w(\cdot))$ *is a stable externally generated past trajectory of* $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$, *then* (8.3) *holds*. This follows from steps 5 and 7. □

Definition 8.2. We call the system $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$ the *canonical model* of a passive controllable forward conservative s/s system with full behavior $\mathfrak{W}$.

In Theorem 2.11 we established the connections (2.15)–(2.17) between passive past, future, and full behaviors $\mathfrak{W}_- = \mathfrak{W}_-$, $\mathfrak{W}_+ = \mathfrak{W}_+$, and $\mathfrak{W} = \mathfrak{W}$ (see Remark 2.13). In particular, they permit us to define unique full behavior $\mathfrak{W}$ in terms of a given past behavior $\mathfrak{W}_-$. Once we have the full behavior $\mathfrak{W}$, we can also define the past/future map $\Gamma_{\mathfrak{W}}$ by (6.1).

Lemma 8.3. *The input map of* $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$ *is the identity on* $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$, *and the output map of* $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$ *is the past/future map* $\Gamma_{\mathfrak{W}}$ *of* $\mathfrak{W}$.

Proof. It follows from Lemma 5.12 and Theorem 8.1 that the $\mathfrak{B}_{\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}}$ acts as the identity on $\mathcal{H}^0(\mathfrak{W}_-^{[\perp]})$, and since $\mathcal{H}^0(\mathfrak{W}_-^{[\perp]})$ is dense in $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$, this means that $\mathfrak{B}_{\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}}$ is the identity. This implies that the output map of $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$ is $\Gamma_{\mathfrak{W}}$, since the product of the input and output maps must be equal to $\Gamma_{\mathfrak{W}}$. □

Corollary 8.4. *The system* $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$ *is both* $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$*-exactly controllable and constructable* (*observable in backward time*), *i.e., if the signal part* $w(\cdot)$ *of a past stable trajectory* $(x(\cdot), w(\cdot))$ *of* $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$ *is zero, then also the state part* $x(\cdot)$ *is zero.*

Proof. The first claim follows from Lemma 8.3 and the second claim follows from (8.4). □

Lemma 8.5. *The pair of sequences* $(x(\cdot), w_+(\cdot))$ *is a stable future trajectory of* $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$ *if and only if*

$$\begin{aligned} w_+ &\in \mathcal{K}(\mathfrak{W}_{\mathrm{fut}}), \\ w_+ + \mathfrak{W}_{\mathrm{fut}} &= \Gamma_{\mathfrak{W}}\big(w_- + \mathfrak{W}_-^{[\perp]}\big), \\ x(n) &= \pi_- S^{-n}(w_- + w_+) + \mathfrak{W}_-^{[\perp]}, \quad n \in \mathbb{Z}^+, \end{aligned} \tag{8.5}$$

for some sequence $w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$ (*in particular,* $x(0) = w_- + \mathfrak{W}_-^{[\perp]}$).

Proof. This follows from Lemma 6.9, taking into account that the input map of $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$ is the identity on $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$. □

Lemma 8.5 gives us the following description of the generating subspace $V_{\mathrm{cfc}}^{\mathfrak{W}_-}$ of $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$:

Theorem 8.6. *Let* $\mathfrak{W}_-$ *be a passive past behavior on the Kreĭn space* $\mathcal{W}$. *Then the generating subspace* $V_{\mathrm{cfc}}^{\mathfrak{W}_-}$ *of the canonical model* $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-} = (V_{\mathrm{cfc}}^{\mathfrak{W}_-}; \mathcal{X}_{\mathrm{cfc}}^{\mathfrak{W}_-}, \mathcal{W})$ *of a passive controllable forward conservative realization of* $\mathfrak{W}_-$ *is given by*

$$V_{\mathrm{cfc}}^{\mathfrak{W}_-}=\left\{\begin{bmatrix}\pi_-S^{-1}w+\mathfrak{W}_-^{[\perp]}\\ \pi_-w+\mathfrak{W}_-^{[\perp]}\\ w(0)\end{bmatrix}\;\middle|\;\begin{array}{l}w=w_-+w_+,\ w_-\in\mathcal{K}\big(\mathfrak{W}_-^{[\perp]}\big),\ w_+\in\mathcal{K}(\mathfrak{W}_+),\\ \text{and } w_++\mathfrak{W}_+=\Gamma_{\mathfrak{W}}\big(w_-+\mathfrak{W}_-^{[\perp]}\big)\end{array}\right\}. \tag{8.6}$$

Proof. This follows from Lemma 8.5 and the fact that $\left[\begin{smallmatrix}x_1\\ x_0\\ w_0\end{smallmatrix}\right]\in V_{\mathrm{cfc}}^{\mathfrak{W}_-}$ if and only if there exists a stable future trajectory $(x(\cdot),w(\cdot))$ of $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$ with $x(0)=x_0$, $x(1)=x_1$, and $w(0)=w_0$. □

Theorem 8.7. *Let $\Sigma=(V;\mathcal{X},\mathcal{W})$ be a passive s/s system with input map $\mathfrak{B}_\Sigma$ and full behavior $\mathfrak{W}$. Then $\mathfrak{B}_\Sigma$ intertwines $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}=(V_{\mathrm{cfc}}^{\mathfrak{W}_-};\mathcal{X}_{\mathrm{cfc}}^{\mathfrak{W}_-},\mathcal{W})$ with Σ in the sense that*

$$\begin{bmatrix}\mathfrak{B}_\Sigma & 0 & 0\\ 0 & \mathfrak{B}_\Sigma & 0\\ 0 & 0 & 1_{\mathcal{W}}\end{bmatrix}V_{\mathrm{cfc}}^{\mathfrak{W}_-}=V\cap\begin{bmatrix}\mathcal{X}\\ \mathcal{R}(\mathfrak{B}_\Sigma)\\ \mathcal{W}\end{bmatrix}. \tag{8.7}$$

Proof. This follows from Lemmas 6.9, 7.7, and 8.5. □

Corollary 8.8. *Any two controllable and forward conservative realizations of a given past behavior $\mathfrak{W}_-$ are unitarily similar to each other.*

Proof. This is true, because by Lemma 5.20, the input maps of these two systems are unitary, and hence, by Theorem 8.7, both systems are unitarily similar to $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$. □

Theorem 8.9. *The operator $\Gamma_{\mathfrak{W}}$ intertwines the two s/s systems $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$ and $\Sigma_{\mathrm{obc}}^{\mathfrak{W}_+}$, i.e.,*

$$\begin{bmatrix}\Gamma_{\mathfrak{W}} & 0 & 0\\ 0 & \Gamma_{\mathfrak{W}} & 0\\ 0 & 0 & 1_{\mathcal{W}}\end{bmatrix}V_{\mathrm{cfc}}^{\mathfrak{W}_-}=V_{\mathrm{obc}}^{\mathfrak{W}_+}\cap\begin{bmatrix}\mathcal{X}_{\mathrm{obc}}^{\mathfrak{W}_+}\\ \mathcal{R}(\Gamma_{\mathfrak{W}})\\ \mathcal{W}\end{bmatrix}. \tag{8.8}$$

Proof. This follows from Theorem 7.8 and Lemma 7.4, and also from Theorem 8.7 and Lemma 8.3. □

The orthogonal companion of $V_{\mathrm{cfc}}^{\mathfrak{W}_-}$ can be characterized as follows:

Lemma 8.10. *The orthogonal companion of $V_{\mathrm{cfc}}^{\mathfrak{W}_-}$ is given by*

$$\big(\mathring{V}_{\mathrm{cfc}}^{\mathfrak{W}_-}\big)^{[\perp]}=\big(V_{\mathrm{cfc}}^{\mathfrak{W}_-}\big)^{[\perp]}=\left\{\begin{bmatrix}w_-+\mathfrak{W}_-^{[\perp]}\\ S_-w_-+\mathfrak{W}_-^{[\perp]}\\ w_-(-1)\end{bmatrix}\;\middle|\;w_-\in\mathcal{K}\big(\mathfrak{W}_-^{[\perp]}\big)\right\}. \tag{8.9}$$

Proof. The proof of this lemma is analogous to step 2 in the proof of Theorem 7.1 which shows that $(\mathring{V}_{\mathrm{obc}})^{[\perp]}=V_{\mathrm{obc}}^{\mathfrak{W}_+}$, where $\mathring{V}_{\mathrm{obc}}$ is the subspace of $\mathfrak{K}_+$ defined in (7.3). We leave this proof to the reader (interchange the first two components in $\mathring{V}_{\mathrm{obc}}$ with each other, replace $\mathfrak{W}_+$ by $-\mathfrak{W}_-^{[\perp]}$, replace $\mathbb{Z}^+$ by $\mathbb{Z}^-$, and replace S_+ by S_-^*). □

9. Frequency domain versions of passive behaviors

The Fourier transform

Up to now we have throughout worked in the *time domain*, and formulated all our results in terms of sequences in $k^2(I;\mathcal{W})$, where I is a discrete time interval. It is also possible to work in the frequency domain instead, replacing all the signal sequences $w(\cdot)$ by their Fourier transforms. In this section we assume, for simplicity, that the signal space $\mathcal{W}$ is separable.

As is well known, for each Hilbert space $\mathcal{X}$, the Fourier transform $\mathcal{F}$, formally defined by $(\mathcal{F}w(\cdot))(z) := \hat{w}(z) = \sum_{n=-\infty}^{\infty} w(n)z^n$, $w(\cdot) \to \hat{w}(\cdot)$ is a unitary map from $\ell^2(\mathcal{X})$ onto the Lebesgue space $L^2(\mathcal{X}) := L^2(\mathbb{T};\mathcal{X})$. The restrictions $\mathcal{F}_\pm = \mathcal{F}|_{\ell^2_\pm(\mathcal{X})}$ of $\mathcal{F}$ to $\ell^2_\pm(\mathcal{X})$ are unitary maps of from $\ell^2_\pm(\mathcal{X})$ onto the Hardy spaces $H^2(\mathbb{D}_\pm;\mathcal{X})$, where

$$\begin{aligned}
\mathbb{D}_+ &:= \bigl\{z \in \mathbb{Z} \bigm| |z| < 1\bigr\},\\
\mathbb{D}_- &:= \bigl\{z \in \mathbb{Z} \bigm| |z| > 1\bigr\} \cup \{\infty\},\\
\mathbb{T} &:= \bigl\{z \in \mathbb{Z} \bigm| |z| = 1\bigr\}.
\end{aligned}$$

Functions in $H^2_\pm(\mathcal{X})$ are analytic in $\mathbb{D}_\pm$, they have boundary values a.e. on $\mathbb{T}$, $L^2(\mathcal{X}) = H^2_-(\mathcal{X}) \oplus H^2_-(\mathcal{X})$, and the norm in these three spaces are given by the same formula

$$\bigl\|\hat{w}(\cdot)\bigr\|^2_{L^2(\mathcal{X})} = \frac{1}{2\pi}\oint_{\zeta\in\mathbb{T}} \bigl\|\hat{w}(\zeta)\bigr\|^2_{\mathcal{X}}\,|\mathrm{d}\zeta|. \tag{9.1}$$

Constant $\mathcal{X}$-valued functions belong to $H^2_+(\mathcal{X})$, and every $\hat{w} \in H^2_-(\mathcal{X})$ satisfies $\hat{w}(\infty) = 0$. The inverse Fourier transform is given by

$$w(n) = \frac{1}{2\pi i}\oint_{\zeta\in\mathbb{T}} \zeta^{-n-1}\hat{w}(\zeta)\,\mathrm{d}\zeta, \quad n \in \mathbb{Z}. \tag{9.2}$$

If $w \in \ell^2_+(\mathcal{X})$ so that $\hat{w} \in H^2_+(\mathcal{X})$, and if $n \in \mathbb{Z}^+$, then

$$w(n) = \frac{\hat{w}^{(n)}(0)}{n!}, \quad n \in \mathbb{Z}^+. \tag{9.3}$$

A similar formula is valid when $w \in \ell^2_-(\mathcal{X})$ and $n \in \mathbb{Z}^-$, involving derivatives of the function $\hat{w}(1/z)$ at the origin. Since $\ell^2(\mathcal{X}) = \ell^2_-(\mathcal{X}) \oplus \ell^2_+(\mathcal{X})$ also $L^2(\mathcal{X}) = \mathcal{H}^2_-(\mathcal{X}) \oplus \mathcal{H}^2_+(\mathcal{X})$. We denote the orthogonal projections of $L^2(\mathcal{X})$ onto $\mathcal{H}^2_\pm(\mathcal{X})$ by $\hat{\pi}_\pm$. They are explicitly given by

$$\begin{aligned}
(\hat{\pi}_+\hat{w})(z) &= \frac{1}{2\pi i}\oint_{\zeta\in\mathbb{T}} (\zeta - z)^{-1}\hat{w}(\zeta)\,\mathrm{d}\zeta, \quad \hat{w} \in L^2(\mathcal{W}),\ z \in \mathbb{D}_+,\\
(\hat{\pi}_-\hat{w})(z) &= -\frac{1}{2\pi i}\oint_{\zeta\in\mathbb{T}} (\zeta - z)^{-1}\hat{w}(\zeta)\,\mathrm{d}\zeta, \quad \hat{w} \in L^2(\mathcal{W}),\ z \in \mathbb{D}_-.
\end{aligned} \tag{9.4}$$

Above we discussed the situation where $\mathcal{X}$ is a Hilbert space. A corresponding theory applies to the case where $\mathcal{X}$ is replaced by a Kreĭn space $\mathcal{W}$. We denote the images of $k^2_+(\mathcal{W})$, $k^2(\mathcal{W})$, and $k^2_-(\mathcal{W})$ under the Fourier transform by $K^2_+(\mathcal{W}) := K^2(\mathbb{D}_+;\mathcal{W})$, $K^2(\mathcal{W}) := K^2(\mathbb{T};\mathcal{W})$, and $K^2_-(\mathcal{W}) := K^2(\mathbb{D}_-;\mathcal{W})$, respectively, and define the indefinite inner products in these spaces so that the Fourier transform is a unitary operator in each case. This means that, if we fix some admissible Hilbert space inner product in $\mathcal{W}$, then the functions in $K^2_+(\mathcal{W})$, $K^2(\mathcal{W})$, and $K^2_-(\mathcal{W})$ belong to $H^2_+(\mathcal{W})$, $L^2(\mathcal{W})$, and $H^2_-(\mathcal{W})$, respectively, and that these three spaces share the same Kreĭn space inner product

$$\big[\hat{w}_1(\cdot),\hat{w}_2(\cdot)\big]_{K^2(\mathcal{W})} = \frac{1}{2\pi}\oint_{\zeta\in\mathbb{T}} \big[\hat{w}_1(\zeta),\hat{w}_2(\zeta)\big]_{\mathcal{W}}\,|\mathrm{d}\zeta|. \tag{9.5}$$

Every fundamental decomposition $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ of the signal space gives rise to the three fundamental decompositions

$$\begin{aligned}
H^2_+(\mathcal{W}) &= -H^2_+(\mathcal{Y})\,[\dotplus]\,H^2_+(\mathcal{U}),\\
L^2(\mathcal{W}) &= -L^2(\mathcal{Y})\,[\dotplus]\,L^2(\mathcal{U}),\\
H^2_-(\mathcal{W}) &= -H^2_-(\mathcal{Y})\,[\dotplus]\,H^2_-(\mathcal{U}).
\end{aligned}$$

Under the Fourier transform the three shift operators S_+, S, and S_- and their adjoints are mapped into the frequency domain shift operators

$$\begin{aligned}
\widehat{S}_+\hat{w}(z) &:= z\hat{w}(z), \quad \hat{w}(\cdot)\in K^2_+(\mathcal{W}),\\
\widehat{S}^*_+\hat{w}(z) &:= \frac{1}{z}\big(\hat{w}(z)-\hat{w}(0)\big), \quad \hat{w}(\cdot)\in K^2_+(\mathcal{W}),\\
\widehat{S}\,\hat{w}(z) &:= z\hat{w}(z), \quad \hat{w}(\cdot)\in K^2(\mathcal{W}),\\
\widehat{S}^{-1}\hat{w}(z) &:= \frac{1}{z}\hat{w}(z), \quad \hat{w}(\cdot)\in K^2(\mathcal{W}),\\
\widehat{S}_-\hat{w}(z) &:= z\hat{w}(z) - \lim_{\zeta\to\infty}\zeta\,\hat{w}(\zeta), \quad \hat{w}(\cdot)\in K^2_-(\mathcal{W}),\\
\widehat{S}^*_-\hat{w}(z) &:= \frac{1}{z}\hat{w}(z), \quad \hat{w}(\cdot)\in K^2_-(\mathcal{W}).
\end{aligned} \tag{9.6}$$

Frequency domain behaviors

Under the Fourier transform the class of all passive future behaviors $\mathfrak{W}_+$ on $\mathcal{W}$ is mapped onto the class of all maximal nonnegative $\widehat{S}_+$-invariant subspaces $\widehat{\mathfrak{W}}_+$ of $K^2_+(\mathcal{W})$, the class of all passive past behaviors $\mathfrak{W}_-$ on $\mathcal{W}$ is mapped onto the class of all maximal nonnegative $\widehat{S}_-$-invariant subspaces $\widehat{\mathfrak{W}}_-$ of $K^2_-(\mathcal{W})$, and the class of all passive full behaviors $\mathfrak{W}$ is mapped onto the class of all maximal nonnegative $\widehat{S}$-reducing causal subspaces $\widehat{\mathfrak{W}}$ of $K^2(\mathcal{W})$. The definition of causality in the frequency domain is analogous to the definition of causality in time domain, i.e., a S-reducing maximal nonnegative subspace $\widehat{\mathfrak{W}}$ is causal if it is true for some fundamental decomposition $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ of $\mathcal{W}$ that

$$\hat{w}(\cdot) \in \widehat{\mathfrak{W}} \quad \text{and} \quad P_{H^2_-(\mathcal{U})}\hat{w} = 0 \quad \Rightarrow \quad \hat{\pi}_-\hat{w}(\cdot) = 0. \tag{9.7}$$

The frequency domain analogue of the space $\mathcal{H}(\mathfrak{W}_+)$ is the Hilbert space $\mathcal{H}(\widehat{\mathfrak{W}}_+)$, where $\widehat{\mathfrak{W}}_+$ is a maximal nonnegative $\widehat{S}_+$-invariant subspace of $K^2_+(\mathcal{W})$, and the frequency domain analogue of the space $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ is the Hilbert space $\mathcal{H}(\widehat{\mathfrak{W}}_-^{[\perp]})$, where $\widehat{\mathfrak{W}}_-$ is a maximal nonnegative $\widehat{S}_-$-invariant subspace of $K^2_-(\mathcal{W})$. These spaces are defined in the same way as in Section 4, with $k^2_\pm(\mathcal{W})$ replaced by $K^2_\pm(\mathcal{W})$ and with $\mathfrak{M}_\pm$ replaced by $\widehat{\mathfrak{W}}_\pm$. Since the $\mathcal{F}_\pm$ are unitary maps of $k^2_\pm(\mathcal{W})$ onto $H^2_\pm(\mathcal{W})$, and since the frequency domain constructions are identical to the time domain constructions, the Fourier transform induces two unitary maps $\mathcal{H}(\mathfrak{W}_\pm) \to \mathcal{H}(\widehat{\mathfrak{W}}_\pm)$ which map $\mathcal{H}^0(\mathfrak{W}_\pm)$ isometrically onto $\mathcal{H}^0(\widehat{\mathfrak{W}}_\pm)$. We shall use the same notation $\mathcal{F}_\pm$ for these two unitary maps.

Given a passive full behavior $\mathfrak{W}$ we define the frequency domain version of the past/future maps of $\mathfrak{W}$ by $\Gamma_{\widehat{\mathfrak{W}}} = \mathcal{F}_+\Gamma_{\mathfrak{W}}\mathcal{F}_-^{-1}$. Thus, if $\mathfrak{W}$ is a passive full behavior on $\mathcal{W}$ with the corresponding passive future and past behaviors $\mathfrak{W}_+$ and $\mathfrak{W}_-$, then $\Gamma_{\widehat{\mathfrak{W}}}$ is the unique linear contraction $\mathcal{H}(\widehat{\mathfrak{W}}_-^{[\perp]}) \to \mathcal{H}(\widehat{\mathfrak{W}}_+)$, which is defined by the relation

$$\hat{\pi}_+\hat{w} + \widehat{\mathfrak{W}}_+ = \Gamma_{\widehat{\mathfrak{W}}}\big(\hat{\pi}_-\hat{w} + \widehat{\mathfrak{W}}_-^{[\perp]}\big), \quad \hat{w} \in \widehat{\mathfrak{W}},$$

on the dense subspace $\mathcal{H}^0(\widehat{\mathfrak{W}}_-^{[\perp]}) := \{\hat{w}_- + \widehat{\mathfrak{W}}_-[\perp] \mid \hat{w}_- \in \widehat{\mathfrak{W}}_-\}$ of $\mathcal{H}(\widehat{\mathfrak{W}}_-^{[\perp]})$ and then extended to $\mathcal{H}(\widehat{\mathfrak{W}}_-^{[\perp]})$ by continuity.

Graph representations of frequency domain behaviors

We next develop graph representations of $\widehat{\mathfrak{W}}$, $\widehat{\mathfrak{W}}_+$, and $\widehat{\mathfrak{W}}_-$ by using the graph representations (2.12)–(2.14) of $\mathfrak{W}_+$, $\mathfrak{W}$, and $\mathfrak{W}_-$. As is well known and easy to prove, the operators $\mathfrak{D}_+$, $\mathfrak{D}$, and $\mathfrak{D}_-$ in appearing in (2.12)–(2.14) have the expansions

$$(\mathfrak{D}_+w_+)(n) = \sum_{k=0}^{n} D(n-k)w(k), \quad w_+ \in k^2_+(\mathcal{U}),\ n \in \mathbb{Z}^+, \tag{9.8}$$

$$(\mathfrak{D}w)(n) = \sum_{k=-\infty}^{n} D(n-k)w(k), \quad w \in k^2(\mathcal{U}),\ n \in \mathbb{Z}, \tag{9.9}$$

$$(\mathfrak{D}_-w_-)(n) = \sum_{k=-\infty}^{n} D(n-k)w(k), \quad w_- \in k^2_-(\mathcal{U}),\ n \in \mathbb{Z}^-, \tag{9.10}$$

with the same coefficients $D(k)$, $k \in \mathbb{Z}^+$, in all the three formulas. If we define $\Phi(z)$ by

$$\Phi(z) = \sum_{n=0}^{\infty} D(n)z^n, \tag{9.11}$$

then Φ is a Schur class function in the unit disk $\mathbb{D}_+$, i.e., a $\mathcal{B}(\mathcal{U},\mathcal{Y})$-valued analytic contractive function in $\mathbb{D}_+$. The radial limits

$$\Phi(\zeta) = \lim_{r\uparrow 1} \Phi(r\zeta), \quad \zeta \in \mathbb{T}, \tag{9.12}$$

exist in the strong sense a.e. on $\mathbb{T}$. The frequency domain analogues of the three operators $\mathfrak{D}$, $\mathfrak{D}_+$, and $\mathfrak{D}_-$ are $\widehat{\mathfrak{D}} = \mathcal{F}\mathfrak{D}\mathcal{F}^{-1}$, $\widehat{\mathfrak{D}}_+ = \widehat{\mathfrak{D}}|_{H^2_+(\mathcal{U})}$, and $\widehat{\mathfrak{D}}_- = \hat{\pi}_+\widehat{\mathfrak{D}}|_{H^2_-(\mathcal{U})}$. Here $\widehat{\mathfrak{D}}$ is a Laurent operator (multiplication operator) with symbol Φ, and $\widehat{\mathfrak{D}}_+$, and $\widehat{\mathfrak{D}}_-$ are the appropriate compressions of $\widehat{\mathfrak{D}}$, i.e.,

$$\begin{aligned}
(\widehat{\mathfrak{D}}\hat{w})(\zeta) &= \Phi(\zeta)\hat{w}(\zeta), \quad \hat{w} \in L^2(\mathcal{W}),\ \zeta \in \mathbb{T}, \\
(\widehat{\mathfrak{D}}_+\hat{w}_+)(z) &= \Phi(z)\hat{w}_+(z), \quad \hat{w}_+ \in H^2_+(\mathcal{W}),\ z \in \mathbb{D}_+, \\
(\widehat{\mathfrak{D}}_-\hat{w}_-)(z) &= -\frac{1}{2\pi i}\oint_{\zeta\in\mathbb{T}} \frac{\Phi(\zeta)\hat{w}_-(\zeta)}{\zeta - z}\,\mathrm{d}\zeta, \quad \hat{w}_- \in H^2_-(\mathcal{W}),\ z \in \mathbb{D}_-.
\end{aligned} \tag{9.13}$$

The adjoint $\widehat{\mathfrak{D}}^*$ of $\widehat{\mathfrak{D}}$ is the Laurent operator whose symbol is $\Phi^*(\zeta)$, $\zeta \in \mathbb{T}$, and $\widehat{\mathfrak{D}}^*_+$ and $\widehat{\mathfrak{D}}^*_+$ are the appropriate compressions of $\widehat{\mathfrak{D}}^*$. The symbol $\Phi^*(\zeta)$ is the radial boundary value of the function $\Phi^*(1/\bar{z})$, $z \in \mathbb{D}_-$, which is a Schur class function in $\mathbb{D}_-$. In terms of the three operators $\widehat{\mathfrak{D}}$ and $\widehat{\mathfrak{D}}_\pm$ the Fourier images $\widehat{\mathfrak{W}} := \mathcal{F}\mathfrak{W}$ and $\widehat{\mathfrak{W}}_\pm := \mathcal{F}_\pm\mathfrak{W}_\pm$ of $\mathfrak{W}$ and $\mathfrak{W}_\pm$ have the graph representations

$$\begin{aligned}
\widehat{\mathfrak{W}} &= \left\{ \hat{w} = \begin{bmatrix} \widehat{\mathfrak{D}}\hat{u} \\ \hat{u} \end{bmatrix} \,\middle|\, \hat{u} \in L^2(\mathcal{U}) \right\}, \\
\widehat{\mathfrak{W}}_\pm &= \left\{ \hat{w}_\pm = \begin{bmatrix} \widehat{\mathfrak{D}}_\pm\hat{u}_\pm \\ \hat{u}_\pm \end{bmatrix} \,\middle|\, \hat{u}_\pm \in H^2_\pm(\mathcal{U}) \right\}.
\end{aligned} \tag{9.14}$$

The de Branges complementary spaces $\mathcal{H}(\widehat{\mathfrak{D}}_+)$ and $\mathcal{H}(\widehat{\mathfrak{D}}^*_-)$

We next describe how the spaces $\mathcal{H}(\widehat{\mathfrak{W}}_+)$ and $\mathcal{H}(\widehat{\mathfrak{W}}^{[\perp]}_-)$ can be mapped unitarily onto the de Branges complementary spaces $\mathcal{H}(\widehat{\mathfrak{D}}_+)$ and $\mathcal{H}(\widehat{\mathfrak{D}}^*_-)$.

The most important fact in the construction of $\mathcal{H}(\widehat{\mathfrak{D}}_+)$ and $\mathcal{H}(\widehat{\mathfrak{D}}^*_-)$ is that both of the operators $\widehat{\mathfrak{D}}_+$ and $\widehat{\mathfrak{D}}^*_-$ are contractions, and below we describe how one constructs the de Branges complementary space $\mathcal{H}(A)$ for a given contraction $A : \widetilde{\mathcal{U}} \to \widetilde{\mathcal{Y}}$ where $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{Y}}$ are Hilbert spaces. This space is defined by the formulas

$$\mathcal{H}(A) = \left\{ \tilde{y} \in \widetilde{\mathcal{Y}} \,\middle|\, \|\tilde{y}\|_{\mathcal{H}(A)} < \infty \right\}, \tag{9.15}$$

where

$$\|\tilde{y}\|_{\mathcal{H}(A)} = \sup\left\{ \|\tilde{y} - A\tilde{u}\|^2_{\widetilde{\mathcal{Y}}} - \|\tilde{u}\|^2_{\widetilde{\mathcal{U}}} \,\middle|\, \tilde{u} \in \widetilde{\mathcal{U}} \right\}. \tag{9.16}$$

This is a Hilbert space continuously contained in $\widetilde{\mathcal{Y}}$. It was introduced and used in [7,8] with A replaced by $\widehat{\mathfrak{D}}_+$ as the state space in the canonical de Branges–Rovnyak model of a scattering i/s/o observable backward conservative system with a given Schur class scattering matrix Φ. We shall derive this model from our s/s model in the next section.

Later it was observed that $\mathcal{H}(A)$ has another alternative characterization:

$$\begin{aligned} \mathcal{H}(A) &= \mathcal{R}\big((1-AA^*)^{1/2}\big), \\ \|\tilde{y}\|_{\mathcal{H}(A)} &= \big\|\big[(1-AA^*)^{1/2}\big]^{[-1]}\tilde{y}\big\|_{\widetilde{\mathcal{Y}}}, \quad \tilde{y} \in \mathcal{H}(A), \end{aligned} \tag{9.17}$$

where the upper index $^{[-1]}$ represents a pseudo-inverse, i.e., $B^{[-1]} : \mathcal{R}(B) \to (\mathcal{N}(B))^\perp$ is the inverse of the injective operator $B|_{(\mathcal{N}(B))^\perp} \to \mathcal{R}(B)$. The operator $(1-AA^*)^{1/2}$ is usually called the *defect operator* of the contraction A^*. See [1,11] for more details.

In [6] it was explained how the space $\mathcal{H}(\mathcal{Z})$ described in Section 4 is related to the space $\mathcal{H}(A)$, where A is the contraction appearing in the graph representation

$$\mathcal{Z} = \left\{ \begin{bmatrix} A\tilde{u} \\ \tilde{u} \end{bmatrix} \,\middle|\, \tilde{u} \in \widetilde{\mathcal{U}} \right\}$$

of the maximal nonnegative subspace $\mathcal{Z}$ of $\mathcal{K}$ with respect to some fundamental decomposition $\mathcal{K} = -\widetilde{\mathcal{Y}} \,[\dot{+}]\, \widetilde{\mathcal{U}}$. The connection is the following. There exists a unitary map $T : \mathcal{H}(\mathcal{Z}) \to \mathcal{H}(A)$ with the property that the image of $x + \mathcal{Z} \in \mathcal{H}(\mathcal{Z})$ under T is the unique vector $\tilde{y}$ in this equivalence class whose projection onto $\widetilde{\mathcal{U}}$ is zero. Explicitly this means that

$$\begin{aligned} T\left(\begin{bmatrix} \tilde{y} \\ \tilde{u} \end{bmatrix} + \mathcal{Z}\right) &= \tilde{y} - A\tilde{u}, \quad \begin{bmatrix} \tilde{y} \\ \tilde{u} \end{bmatrix} \in \mathcal{K}(\mathcal{Z}), \\ T^{-1}\tilde{y} &= \begin{bmatrix} \tilde{y} \\ 0 \end{bmatrix} + \mathcal{Z}, \quad \tilde{y} \in \mathcal{H}(A). \end{aligned} \tag{9.18}$$

The operator T maps $\mathcal{H}^0(\mathcal{Z})$ one-to-one onto the dense subspace $\mathcal{R}(1-AA^*)$ of $\mathcal{H}(A)$. In the sequel we denote $\mathcal{H}^0(A) := \mathcal{R}(1-AA^*)$.

We now apply the theory described above with the following alternative replacements:

(1) $\mathcal{Z} = \widehat{\mathfrak{W}}_+$, $A = \widehat{\mathfrak{D}}_+$, $\widetilde{\mathcal{U}} = H^2_+(\mathcal{U})$, $\widetilde{\mathcal{Y}} = H^2_+(\mathcal{Y})$, and $T = \widehat{T}_+$,
(2) $\mathcal{Z} = \widehat{\mathfrak{W}}_-^{[\perp]}$, $A = \widehat{\mathfrak{D}}_-^*$, $\widetilde{\mathcal{U}} = H^2_-(\mathcal{Y})$, $\widetilde{\mathcal{Y}} = H^2_-(\mathcal{U})$, and $T = \widehat{T}_-$.

We leave it to the reader to carry out these substitutions in (9.15)–(9.17). When we do the same substitution in (9.18) we get

$$\begin{aligned} \widehat{T}_+\left(\begin{bmatrix} \hat{y}_+ \\ \hat{u}_+ \end{bmatrix} + \widehat{\mathfrak{W}}_+\right) &= \hat{y}_+ - \widehat{\mathfrak{D}}_+\hat{u}_+, \quad \begin{bmatrix} \hat{y}_+ \\ \hat{u}_+ \end{bmatrix} \in \mathcal{K}(\widehat{\mathfrak{W}}_+), \\ \widehat{T}_-\left(\begin{bmatrix} \hat{y}_- \\ \hat{u}_- \end{bmatrix} + \widehat{\mathfrak{W}}_-^{[\perp]}\right) &= \hat{u}_- - \widehat{\mathfrak{D}}_-^*\hat{y}_-, \quad \begin{bmatrix} \hat{y}_- \\ \hat{u}_- \end{bmatrix} \in \mathcal{K}\big(\widehat{\mathfrak{W}}_-^{[\perp]}\big), \\ \widehat{T}_+^{-1}\hat{y}_+ &= \begin{bmatrix} \hat{y}_+ \\ 0 \end{bmatrix} + \widehat{\mathfrak{W}}_+, \quad \hat{y}_+ \in \mathcal{H}(\mathfrak{W}_+), \\ \widehat{T}_-^{-1}\hat{u}_- &= \begin{bmatrix} 0 \\ \hat{u}_- \end{bmatrix} + \widehat{\mathfrak{W}}_-^{[\perp]}, \quad \hat{u}_- \in \mathcal{H}\big(\mathfrak{W}_-^{[\perp]}\big). \end{aligned} \tag{9.19}$$

The past/future map from $\mathcal{H}(\widehat{\mathfrak{D}}_-^*)$ to $\mathcal{H}(\widehat{\mathfrak{D}}_+)$

By using the unitary maps $\widehat{T}_- : \mathcal{H}(\widehat{\mathfrak{W}}_-^{[\perp]}) \to \mathcal{H}(\widehat{\mathfrak{D}}_-^*)$ and $\widehat{T}_+ : \mathcal{H}(\widehat{\mathfrak{W}}_+) \to \mathcal{H}(\widehat{\mathfrak{D}}_+)$ we can define a version of the past/future map of a passive full behavior which is a contraction from $\mathcal{H}(\widehat{\mathfrak{D}}_-^*)$ to $\mathcal{H}(\widehat{\mathfrak{D}}_+)$, namely

$$\Gamma_{(\widehat{\mathfrak{D}}_-^*, \widehat{\mathfrak{D}}_+)} := \widehat{T}_+ \Gamma_{\widehat{\mathfrak{W}}} \widehat{T}_-^{-1} = \widehat{T}_+ \mathcal{F}_+ \Gamma_{\mathfrak{W}} \mathcal{F}_-^{-1} \widehat{T}_-^{-1}.$$

This map is related to but not identical with the *Hankel operator*

$$\Gamma_{\widehat{\mathfrak{D}}} := \hat{\pi}_+ \widehat{\mathfrak{D}} \hat{\pi}_- : H_-^2(\mathcal{U}) \to H_+^2(\mathcal{Y})$$

induced by $\widehat{\mathfrak{D}}$. Before we explaining the exact connection we first prove the following lemma.

Lemma 9.1. *Let $\hat{w} \in \widehat{\mathfrak{W}}$, and write $\hat{w}$ in the form $\hat{w} = \left[\begin{smallmatrix} \widehat{\mathfrak{D}}\hat{u} \\ \hat{u} \end{smallmatrix}\right]$ where $\hat{u} = P_{L^2(\mathcal{U})}\hat{u} \in L^2(\mathcal{U})$* (*cf.* (9.14)). *Then*

$$\begin{aligned} \widehat{T}_-\big(\hat{\pi}_- \hat{w} + \widehat{\mathfrak{W}}_-^{[\perp]}\big) &= \big(1 - \widehat{\mathfrak{D}}_-^* \widehat{\mathfrak{D}}_-\big)\hat{u}_-, \\ \widehat{T}_+(\hat{\pi}_+ \hat{w} + \widehat{\mathfrak{W}}_+) &= \Gamma_{\widehat{\mathfrak{D}}} \hat{u}_-, \end{aligned} \tag{9.20}$$

where $\hat{u}_- = \hat{\pi}_- u \in H_-^2(\mathcal{U})$.

Proof. Since $\hat{\pi}_- w = \left[\begin{smallmatrix} \hat{\pi}_- \widehat{\mathfrak{D}}\hat{u} \\ \hat{\pi}_- \hat{u} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \widehat{\mathfrak{D}}_- \hat{u}_- \\ \hat{u}_- \end{smallmatrix}\right]$, we get from (9.19),

$$\widehat{T}_-\big(\hat{\pi}_- \hat{w} + \widehat{\mathfrak{W}}_-^{[\perp]}\big) = \hat{u}_- + \widehat{\mathfrak{D}}_-^*(\widehat{\mathfrak{D}}_- \hat{u}_-) = \big(1 - \widehat{\mathfrak{D}}_-^* \widehat{\mathfrak{D}}_-\big)\hat{u}_-,$$

which is the first claim in (9.20). Analogously,

$$\hat{\pi}_+ \hat{w} = \begin{bmatrix} \hat{\pi}_+ \widehat{\mathfrak{D}} \hat{u} \\ \hat{\pi}_+ \hat{u} \end{bmatrix} = \begin{bmatrix} \widehat{\mathfrak{D}}_+ \hat{u}_+ \\ \hat{u}_+ \end{bmatrix} + \begin{bmatrix} \Gamma_{\widehat{\mathfrak{D}}} \hat{u}_- \\ 0 \end{bmatrix}, \tag{9.21}$$

where $\hat{u}_+ = \hat{\pi}_+ \hat{u} \in H_+^2(\mathcal{Y})$. The first component in the above sum belongs to $\widehat{\mathfrak{W}}_+$, and hence by (9.19), $\widehat{T}_+(\hat{\pi}_+ \hat{w} + \widehat{\mathfrak{W}}_+) = \Gamma_{\widehat{\mathfrak{D}}} \hat{u}_-$. $\square$

Lemma 9.2. *The operator $\Gamma_{(\widehat{\mathfrak{D}}_-^*, \widehat{\mathfrak{D}}_+)}$ is the unique linear contraction $\mathcal{H}(\widehat{\mathfrak{D}}_-^*) \to \mathcal{H}(\widehat{\mathfrak{D}}_+)$, which is defined by the relation*

$$\Gamma_{(\widehat{\mathfrak{D}}_-^*, \widehat{\mathfrak{D}}_+)} = \Gamma_{\widehat{\mathfrak{D}}} \big(1 - \widehat{\mathfrak{D}}_-^* \widehat{\mathfrak{D}}_-\big)^{[-1]}, \tag{9.22}$$

on the dense subspace $\mathcal{H}^0(\widehat{\mathfrak{D}}_-^*) = \mathcal{R}(1 - \widehat{\mathfrak{D}}_-^* \widehat{\mathfrak{D}}_-)$ of $\mathcal{H}(\widehat{\mathfrak{D}}_-^*)$ and then extended to $\mathcal{H}(\widehat{\mathfrak{D}}_-^*)$ by continuity.

Proof. By the Fourier transformed version of Lemma 6.1,

$$\Gamma_{\widehat{\mathfrak{W}}}\big(\hat{\pi}_-\hat{w} + \widehat{\mathfrak{W}}_-^{[\perp]}\big) = \hat{\pi}_+\hat{w} + \widehat{\mathfrak{W}}_+, \quad \hat{w} \in \widehat{\mathfrak{W}}. \tag{9.23}$$

This together with (9.20) gives

$$\Gamma_{\widehat{\mathfrak{D}}}\hat{u}_- = \widehat{T}_+(\hat{\pi}_+\hat{w} + \widehat{\mathfrak{W}}_+) = \widehat{T}_+\Gamma_{\widehat{\mathfrak{W}}}\big(\hat{\pi}_-\hat{w} + \widehat{\mathfrak{W}}_-^{[\perp]}\big) = \widehat{T}_+\Gamma_{\widehat{\mathfrak{W}}}\widehat{T}_-^{-1}\big(1 - \widehat{\mathfrak{D}}_-^*\widehat{\mathfrak{D}}_-\big)\hat{u}_-.$$

Here $\hat{u}_-$ can be an arbitrary function in $H^2_-(\mathcal{U})$, and consequently

$$\Gamma_{\widehat{\mathfrak{D}}} = \Gamma_{(\widehat{\mathfrak{D}}_-^*, \widehat{\mathfrak{D}}_+)}\big(1 - \widehat{\mathfrak{D}}_-^*\widehat{\mathfrak{D}}_-\big). \tag{9.24}$$

By applying the pseudo-inverse $(1 - \widehat{\mathfrak{D}}_-^*\widehat{\mathfrak{D}}_-)^{[-1]}$ to both sides of (9.24) we get the conclusion of Lemma 9.2 $\square$

The operator $1 - \widehat{\mathfrak{D}}_-^*\widehat{\mathfrak{D}}_-$ appearing in Lemma 9.2 has a natural interpretation:

Lemma 9.3. *The adjoint of the inclusion map* $\mathcal{H}(\widehat{\mathfrak{D}}_-^*) \hookrightarrow H^2_-(\mathcal{U})$ *is the operator* $1 - \widehat{\mathfrak{D}}_-^*\widehat{\mathfrak{D}}_- : H^2_-(\mathcal{U}) \to \mathcal{H}(\widehat{\mathfrak{D}}_-^*)$.

Proof. By (9.17), every $\hat{u}_- \in \mathcal{H}(\widehat{\mathfrak{D}}_-^*)$ can be written in the form $\hat{u}_- = (1 - \widehat{\mathfrak{D}}_-^*\widehat{\mathfrak{D}}_-)^{1/2}\hat{u}_0$ for some $\hat{u}_0 \in H^2_-(\mathcal{U})$. Therefore, for every $\hat{u}_1 \in H^2_-(\mathcal{U})$ (to get the third equality sign below we polarize the second identity in (9.17))

$$\begin{aligned}
(\hat{u}_-, \hat{u}_1)_{H^2_-(\mathcal{U})} &= \big(\big(1 - \widehat{\mathfrak{D}}_-^*\widehat{\mathfrak{D}}_-\big)^{1/2}\hat{u}_0, \hat{u}_1\big)_{H^2_-(\mathcal{U})} \\
&= \big(\hat{u}_0, \big(1 - \widehat{\mathfrak{D}}_-^*\widehat{\mathfrak{D}}_-\big)^{1/2}\hat{u}_1\big)_{H^2_-(\mathcal{U})} \\
&= \big(\big(1 - \widehat{\mathfrak{D}}_-^*\widehat{\mathfrak{D}}_-\big)^{1/2}\hat{u}_0, \big(1 - \widehat{\mathfrak{D}}_-^*\widehat{\mathfrak{D}}_-\big)\hat{u}_1\big)_{\mathcal{H}(\widehat{\mathfrak{D}}_-^*)} \\
&= \big(\hat{u}_-, \big(1 - \widehat{\mathfrak{D}}_-^*\widehat{\mathfrak{D}}_-\big)\hat{u}_1\big)_{\mathcal{H}(\widehat{\mathfrak{D}}_-^*)}. \qquad \square
\end{aligned}$$

10. Input/output representations of passive behaviors

Frequency domain versions of the canonical s/s models

By using the Fourier transform we can map the two canonical models $\Sigma_{\mathrm{obc}}^{\mathfrak{W}_+}$ and $\Sigma_{\mathrm{cfc}}^{\mathfrak{W}_-}$ into the frequency domain, to get two canonical frequency domain models $\Sigma_{\mathrm{obc}}^{\widehat{\mathfrak{W}}_+}$ and $\Sigma_{\mathrm{cfc}}^{\widehat{\mathfrak{W}}_-}$ whose frequency domain full behavior is $\widehat{\mathfrak{W}}$. The generating subspace of the frequency domain passive observable and backward conservative model $\widehat{\Sigma}_{\mathrm{obc}}^{\widehat{\mathfrak{W}}_+}$ is given by

$$V_{\mathrm{obc}}^{\widehat{\mathfrak{W}}_+} = \left\{ \begin{bmatrix} \widehat{S}_+^*\hat{w} + \widehat{\mathfrak{W}}_+ \\ \hat{w} + \widehat{\mathfrak{W}}_+ \\ \hat{w}(0) \end{bmatrix} \in \begin{bmatrix} \mathcal{H}(\widehat{\mathfrak{W}}_+) \\ \mathcal{H}(\widehat{\mathfrak{W}}_+) \\ \mathcal{W} \end{bmatrix} \;\middle|\; \hat{w} \in \mathcal{K}(\widehat{\mathfrak{W}}_+) \right\}, \tag{10.1}$$

and the generating subspace of the frequency domain controllable and forward conservative model is

$$V_{\mathrm{cfc}}^{\widehat{\mathfrak{W}}_-}=\left\{\begin{bmatrix}\hat{\pi}_-\widehat{S}^{-1}\hat{w}+\widehat{\mathfrak{W}}_-^{[\perp]}\\ \hat{\pi}_-\hat{w}+\widehat{\mathfrak{W}}_-^{[\perp]}\\ \hat{w}_+(0)\end{bmatrix}\;\middle|\;\begin{array}{l}\hat{w}=\hat{w}_-+\hat{w}_+,\ \hat{w}_-\in\mathcal{K}\big(\widehat{\mathfrak{W}}_-^{[\perp]}\big),\ \hat{w}_+\in\mathcal{K}(\widehat{\mathfrak{W}}_+),\\ \text{and } \hat{w}_++\widehat{\mathfrak{W}}_+=\widehat{\Gamma}_{\widehat{\mathfrak{W}}}\big(\hat{w}_-+\widehat{\mathfrak{W}}_-^{[\perp]}\big)\end{array}\right\}. \tag{10.2}$$

The first canonical de Branges–Rovnyak model

We continue by developing a description of the i/s/o representation of $\Sigma_{\mathrm{obc}}^{\widehat{\mathfrak{W}}_+}$ corresponding to a fundamental decomposition $\mathcal{W}=-\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ of the signal space $\mathcal{W}$. We begin by applying the unitary similarity transform $\widehat{T}_+$ to $\Sigma_{\mathrm{obc}}^{\widehat{\mathfrak{W}}_+}$ in order to replace the state space $\mathcal{H}(\widehat{\mathfrak{W}}_+)$ of $\Sigma_{\mathrm{obc}}^{\widehat{\mathfrak{W}}_+}$ by the state space $\mathcal{H}(\widehat{\mathfrak{D}}_+)$ of the new system $\Sigma_{\mathrm{obc}}^{\widehat{\mathfrak{D}}_+}$ with generating subspace

$$V_{\mathrm{obc}}^{\widehat{\mathfrak{D}}_+}:=\begin{bmatrix}\widehat{T}_+&0&0\\0&\widehat{T}_+&0\\0&0&1_{\mathcal{W}}\end{bmatrix}V_{\mathrm{obc}}^{\widehat{\mathfrak{W}}}.$$

We decompose the parameter $w_+\in\mathcal{K}(\widehat{\mathfrak{W}}_+)$ in (10.1) in the form $\hat{w}_+=\left[\begin{smallmatrix}\hat{y}_+\\ \hat{u}_+\end{smallmatrix}\right]$. Then $\hat{w}_+(0)=\left[\begin{smallmatrix}\hat{y}_+(0)\\ \hat{u}_+(0)\end{smallmatrix}\right]$, and $(\widehat{S}_+^*\hat{w}_+)(z)=\left[\begin{smallmatrix}\widehat{S}_+^*\hat{y}_+\\ \widehat{S}_+^*\hat{u}_+\end{smallmatrix}\right]$. Thus, by (9.19), for all $z\in\mathbb{D}_+$,

$$\begin{aligned}\big(\widehat{T}_+\big(\widehat{S}_+^*\hat{w}_++\widehat{\mathfrak{W}}_+\big)\big)(z)&=\big(\widehat{S}_+^*\hat{y}_+-\widehat{\mathfrak{D}}_+\widehat{S}_+^*\hat{u}_+\big)(z)\\&=\frac{1}{z}\big(\hat{y}_+(z)-\hat{y}_+(0)-\Phi(z)\big(\hat{u}_+(z)-\hat{u}_+(0)\big)\big).\end{aligned}$$

Denoting

$$\begin{aligned}&\hat{x}_0=\widehat{T}_+(\hat{w}_++\widehat{\mathfrak{W}}_+)=\hat{y}_+-\widehat{\mathfrak{D}}_+\hat{u}_+,\\&u_0=\hat{u}_+(0),\end{aligned}$$

and observing that $\hat{x}_0$ can be an arbitrary vector in $\mathcal{H}(\widehat{\mathfrak{D}}_+)$ and u_0 can be an arbitrary vector in $\mathcal{U}$ we get

$$V_{\mathrm{obc}}^{\widehat{\mathfrak{D}}_+}=\left\{\begin{bmatrix}A_{\mathrm{obc}}\hat{x}_0+B_{\mathrm{obc}}u_0\\ \hat{x}_0\\ C_{\mathrm{obc}}\hat{x}_0+C_{\mathrm{obc}}u_0\\ u_0\end{bmatrix}\in\begin{bmatrix}\mathcal{H}(\widehat{\mathfrak{D}}_+)\\ \mathcal{H}(\widehat{\mathfrak{D}}_+)\\ \mathcal{Y}\\ \mathcal{U}\end{bmatrix}\;\middle|\;\hat{x}_0\in\mathcal{H}(\widehat{\mathfrak{D}}_+)\text{ and }u_0\in\mathcal{U}\right\}, \tag{10.3}$$

where

$$
\begin{aligned}
(A_{\mathrm{obc}}\hat{x}_0)(z) &= \frac{1}{z}\big(\hat{x}_0(z)-\hat{x}_0(0)\big), \quad \hat{x}_0 \in \mathcal{H}(\widehat{\mathfrak{D}}_+),\ z\in\mathbb{D}_+,\\
(B_{\mathrm{obc}}u_0)(z) &= \frac{1}{z}\big(\Phi(z)-\Phi(0)\big)u_0, \quad u_0\in\mathcal{U},\ z\in\mathbb{D}_+,\\
C_{\mathrm{obc}}\hat{x}_0 &= \hat{x}_0(0), \quad \hat{x}_0\in\mathcal{H}(\widehat{\mathfrak{D}}_+),\\
D_{\mathrm{obc}} &= \Phi(0).
\end{aligned}
\tag{10.4}
$$

Here $\left[\begin{smallmatrix} A_{\mathrm{obc}} & B_{\mathrm{obc}} \\ C_{\mathrm{obc}} & E_{\mathrm{obc}} \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{H}(\widehat{\mathfrak{D}}_+) \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{H}(\widehat{\mathfrak{D}}_+) \\ \mathcal{Y} \end{smallmatrix}\right]$ is a linear co-isometric operator, and (10.3) is a graph representation of $V_{\mathrm{obc}}^{\widehat{\mathfrak{D}}_+}$ of the type (1.5). Thus, the i/s/o representation

$$
\widehat{\Sigma}_{\mathrm{i/s/o}}^{\widehat{\mathfrak{D}}_+} = \left(\begin{bmatrix} A_{\mathrm{obc}} & B_{\mathrm{obc}} \\ C_{\mathrm{obc}} & D_{\mathrm{obc}} \end{bmatrix}; \mathcal{H}(\widehat{\mathfrak{D}}_+), \mathcal{U}, \mathcal{Y} \right)
$$

of $\Sigma_{\mathrm{obc}}^{\widehat{\mathfrak{D}}_+}$ that we obtain in this way is the canonical de Branges–Rovnyak model of an observable backward conservative scattering system with the scattering matrix Φ mentioned above. This system is observable since $\Sigma_{\mathrm{obc}}^{\widehat{\mathfrak{D}}_+}$ is observable, i.e., $\bigcap_{n\geqslant 0}\mathcal{N}(C_{\mathrm{obc}}A_{\mathrm{obc}}^n)=\{0\}$, and the scattering matrix $zC_{\mathrm{obc}}(1-zA_{\mathrm{obc}})^{-1}B_{\mathrm{obc}}+D_{\mathrm{obc}}$ of this system is equal to $\Phi(z)$.

The second canonical de Branges–Rovnyak model

By applying the unitary similarity transformation $\widehat{T}_-$ to the system $\Sigma_{\mathrm{cfc}}^{\widehat{\mathfrak{W}}_-}$ whose generating subspace is given in (10.2) we get another system $\Sigma_{\mathrm{cfc}}^{\widehat{\mathfrak{D}}_-}$ whose generating subspace is

$$
V_{\mathrm{cfc}}^{\widehat{\mathfrak{D}}_-} := \begin{bmatrix} \widehat{T}_- & 0 & 0 \\ 0 & \widehat{T}_- & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V_{\mathrm{cfc}}^{\widehat{\mathfrak{W}}_-}. \tag{10.5}
$$

This subspace contains the dense subspace

$$
\mathring{V}_{\mathrm{cfc}}^{\widehat{\mathfrak{D}}_-} := \begin{bmatrix} \widehat{T}_- & 0 & 0 \\ 0 & \widehat{T}_- & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} \mathring{V}_{\mathrm{cfc}}^{\widehat{\mathfrak{W}}_-}, \tag{10.6}
$$

where $\mathring{V}_{\mathrm{cfc}}^{\widehat{\mathfrak{W}}_-}$ is the frequency domain version of the subspace $\mathring{V}_{\mathrm{cfc}}^{\mathfrak{W}_-}$ defined in (8.2), i.e.,

$$
\mathring{V}_{\mathrm{cfc}}^{\widehat{\mathfrak{W}}_-} = \left\{ \begin{bmatrix} \hat{\pi}_-\widehat{S}^{-1}\hat{w} + \widehat{\mathfrak{W}}_-^{[\perp]} \\ \hat{\pi}_-\hat{w} + \widehat{\mathfrak{W}}_-^{[\perp]} \\ \hat{w}(0) \end{bmatrix} \in \begin{bmatrix} \mathcal{H}(\widehat{\mathfrak{W}}_-^{[\perp]}) \\ \mathcal{H}(\widehat{\mathfrak{W}}_-^{[\perp]}) \\ \mathcal{W} \end{bmatrix} \;\middle|\; \hat{w}\in\widehat{\mathfrak{W}} \right\}. \tag{10.7}
$$

We parametrize $\hat{w}$ in (10.7) by $\hat{w} = \left[\begin{smallmatrix} \widehat{\mathfrak{D}}\hat{u} \\ \hat{u} \end{smallmatrix}\right]$ where $\hat{u} = P_{L^2(\mathcal{U})}$ is a free parameter in $L^2(\mathcal{U})$, and denote $\hat{u}_\pm = \hat{\pi}_\pm\hat{u}$. By (9.20),

$$
\widehat{T}_-\big(\hat{\pi}_-\hat{w} + \widehat{\mathfrak{W}}_-^{[\perp]}\big) = \big(1-\widehat{\mathfrak{D}}_-^*\widehat{\mathfrak{D}}_-\big)\hat{u}_-.
$$

Recalling that $\widehat{\mathfrak{D}}u = \widehat{\mathfrak{D}}_-\hat{u}_- + \widehat{\mathfrak{D}}_+\hat{u}_+ + \Gamma_{\widehat{\mathfrak{D}}}\hat{u}_-$ and using (9.19) we get

$$\begin{aligned}&\big(\widehat{T}_-\big(\hat{\pi}_-\widehat{S}^{-1}\hat{w} + \widehat{\mathfrak{W}}_-^{[\perp]}\big)\big)(z)\\&\quad = \big(\hat{\pi}_-\widehat{S}^{-1}\hat{u}\big)(z) - \big(\widehat{\mathfrak{D}}_-^*\hat{\pi}_-\widehat{S}^{-1}(\widehat{\mathfrak{D}}_-\hat{u}_- + \widehat{\mathfrak{D}}_+\hat{u}_+ + \Gamma_{\widehat{\mathfrak{D}}}\hat{u}_-)\big)(z)\\&\quad = \frac{1}{z}\big(\big(1 - \widehat{\mathfrak{D}}_-^*\widehat{\mathfrak{D}}_-\big)\hat{u}_-\big)(z) - \frac{1}{z}\Phi(1/\bar{z})\big(\Phi(0)\hat{u}_+(0) + (\Gamma_{\widehat{\mathfrak{D}}}\hat{u}_-)(0)\big).\end{aligned}$$

Denoting

$$\begin{aligned}\hat{x}_0 &= \big(1 - \widehat{\mathfrak{D}}_-^*\widehat{\mathfrak{D}}_-\big)\hat{u}_-,\\ u_0 &= \hat{u}_+(0),\end{aligned}$$

and using Lemma 9.2 and the fact that $\hat{x}_0$ can be an arbitrary vector in $\mathcal{H}^0(\widehat{\mathfrak{D}}_-^*)$ and u_0 can be an arbitrary vector in $\mathcal{U}$ we get

$$\mathring{V}_{\mathrm{cfc}}^{\widehat{\mathfrak{D}}_-^*} = \left\{\begin{bmatrix} A_{\mathrm{cfc}}\hat{x}_0 + B_{\mathrm{cfc}}u_0 \\ \hat{x}_0 \\ C_{\mathrm{cfc}}\hat{x}_0 + D_{\mathrm{cfc}}u_0 \\ u_0\end{bmatrix} \in \begin{bmatrix}\mathcal{H}(\widehat{\mathfrak{D}}_-^*)\\ \mathcal{H}(\widehat{\mathfrak{D}}_-^*)\\ \mathcal{Y}\\ \mathcal{U}\end{bmatrix} \;\middle|\; \hat{x}_0 \in \mathcal{H}^0\big(\widehat{\mathfrak{D}}_-^*\big) \text{ and } u_0 \in \mathcal{U}\right\}, \tag{10.8}$$

where

$$\begin{aligned}(A_{\mathrm{cfc}}\hat{x}_0)(z) &= \frac{1}{z}\big(\hat{x}_0(z) - \Phi^*(1/\bar{z})(\Gamma_{(\widehat{\mathfrak{D}}_-^*,\widehat{\mathfrak{D}}_+)}\hat{x}_0)(0)\big), \quad \hat{x}_0 \in \mathcal{H}\big(\widehat{\mathfrak{D}}_-^*\big),\ z \in \mathbb{D}_-,\\ (B_{\mathrm{cfc}}u_0)(z) &= \frac{1}{z}\big(1_{\mathcal{U}} - \Phi^*(1/\bar{z})\Phi(0)\big)u_0, \quad u_0 \in \mathcal{U},\ z \in \mathbb{D}_-,\\ C_{\mathrm{cfc}}\hat{x}_0 &= (\Gamma_{(\widehat{\mathfrak{D}}_-^*,\widehat{\mathfrak{D}}_+)}\hat{x}_0)(0), \quad \hat{x}_0 \in \mathcal{H}\big(\widehat{\mathfrak{D}}_-^*\big),\\ D_{\mathrm{cfc}} &= \Phi(0).\end{aligned} \tag{10.9}$$

Since $\mathcal{H}^0(\widehat{\mathfrak{D}}_-^*)$ is dense in $\mathcal{H}(\widehat{\mathfrak{D}}_-^*)$ we find that

$$V_{\mathrm{cfc}}^{\widehat{\mathfrak{D}}_-} = \left\{\begin{bmatrix} A_{\mathrm{cfc}}\hat{x}_0 + B_{\mathrm{cfc}}u_0 \\ \hat{x}_0 \\ C_{\mathrm{cfc}}\hat{x}_0 + C_{\mathrm{cfc}}u_0 \\ u_0\end{bmatrix} \in \begin{bmatrix}\mathcal{H}(\widehat{\mathfrak{D}}_-^*)\\ \mathcal{H}(\widehat{\mathfrak{D}}_-^*)\\ \mathcal{Y}\\ \mathcal{U}\end{bmatrix} \;\middle|\; \hat{x}_0 \in \mathcal{H}\big(\widehat{\mathfrak{D}}_-^*\big) \text{ and } u_0 \in \mathcal{U}\right\}. \tag{10.10}$$

Here $\left[\begin{smallmatrix}A_{\mathrm{cfc}} & B_{\mathrm{cfc}}\\ C_{\mathrm{cfc}} & E_{\mathrm{cfc}}\end{smallmatrix}\right] : \left[\begin{smallmatrix}\mathcal{H}(\widehat{\mathfrak{D}}_-^*)\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{H}(\widehat{\mathfrak{D}}_-^*)\\ \mathcal{Y}\end{smallmatrix}\right]$ is an isometric operator, and (10.10) is a graph representation of $V_{\mathrm{cfc}}^{\widehat{\mathfrak{D}}_-}$ of the type (1.5). Thus, the i/s/o representation

$$\Sigma_{\mathrm{i/s/o}}^{\widehat{\mathfrak{D}}_-} = \left(\begin{bmatrix}A_{\mathrm{cfc}} & B_{\mathrm{cfc}}\\ C_{\mathrm{cfc}} & D_{\mathrm{cfc}}\end{bmatrix}; \mathcal{H}\big(\widehat{\mathfrak{D}}_-^*\big), \mathcal{U}, \mathcal{Y}\right)$$

of $\Sigma_{\mathrm{cfc}}^{\widehat{\mathfrak{D}}_+}$ that we obtain in this way is the canonical de Branges–Rovnyak model of a controllable forward conservative scattering system with the scattering matrix Φ. This system is controllable since $\Sigma_{\mathrm{cfc}}^{\widehat{\mathfrak{D}}_-}$ is controllable, i.e., $\bigvee_{n\geqslant 0}\mathcal{R}(A_{\mathrm{cfc}}^n B_{\mathrm{cfc}})=\mathcal{X}$, and the scattering matrix $zC_{\mathrm{cfc}}(1-zA_{\mathrm{cfc}})^{-1}B_{\mathrm{cfc}}+D_{\mathrm{cfc}}$ of this system is equal to $\Phi(z)$.

The formulas for the adjoints of the operators A_{cfc}, B_{cfc}, C_{cfc}, and D_{cfc} in (10.9) are simpler than the formulas for these operators themselves, and they are also easier to compute. This can be done without any knowledge of the past/future map $\Gamma_{(\widehat{\mathfrak{D}}_-^*,\widehat{\mathfrak{D}}_+)}$. Explicitly, these adjoints are given by

$$\begin{aligned}\big(A_{\mathrm{cfc}}^*\hat{x}_0\big)(z) &= z\hat{x}_0(z)-\lim_{\zeta\to\infty}\zeta\hat{x}_0(\zeta), \quad \hat{x}_0\in\mathcal{H}\big(\widehat{\mathfrak{D}}_-^*\big),\ z\in\mathbb{D}_-,\\ B_{\mathrm{cfc}}^*\hat{x}_0 &= \lim_{\zeta\to\infty}\zeta\hat{x}_0(\zeta), \quad \hat{x}_0\in\mathcal{H}\big(\widehat{\mathfrak{D}}_-^*\big),\\ \big(C_{\mathrm{cfc}}^*y_0\big)(z) &= \big(\Phi^*(1/\bar{z})-\Phi^*(0)\big)y_0, \quad y_0\in\mathcal{Y},\ z\in\mathbb{D}_-,\\ D_{\mathrm{cfc}}^* &= \Phi^*(0).\end{aligned} \tag{10.11}$$

The most straightforward way to compute these adjoints is to repeat the computation leading to (10.4) with (7.1) replaced by (8.9), $\mathfrak{W}_+$ replaced by $\mathfrak{W}_-^{[\perp]}$, $\mathfrak{D}_+$ replaced by $\mathfrak{D}_-^*$, and S_+^* replaced by S_-. However, they can, of course, also be computed directly from (10.9). We leave the proof of (10.11) to the reader.

Input and output maps of i/s/o representations

Let $\Sigma=(V;\mathcal{X},\mathcal{W})$ be a passive s/s system, and let $\mathfrak{B}_\Sigma:\mathcal{H}(\mathfrak{W}_-^{[\perp]})\to\mathcal{X}$ and $\mathfrak{C}_\Sigma:\mathcal{X}\to\mathcal{H}(\mathfrak{W}_+)$ be the input and output maps of Σ, where $\mathfrak{W}_-=\mathfrak{W}_{\mathrm{past}}^\Sigma$ and $\mathfrak{W}_+=\mathfrak{W}_{\mathrm{fut}}^\Sigma$ are the past and future behaviors of Σ. We again map $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ unitarily onto $\mathcal{H}(\widehat{\mathfrak{W}}_-^{[\perp]})$ by means of $\mathcal{F}_-$ and $\mathcal{H}(\mathfrak{W}_+)$ unitarily onto $\mathcal{H}(\widehat{\mathfrak{W}}_+)$ by means of $\mathcal{F}_+$. Under these transformations $\mathfrak{B}_\Sigma$ and $\mathfrak{C}_\Sigma$ are mapped onto the frequency domain input and output maps

$$\mathfrak{B}_{\widehat{\Sigma}}=\mathfrak{B}_\Sigma\mathcal{F}_-^{-1}, \qquad \mathfrak{C}_{\widehat{\Sigma}}=\mathcal{F}^+\mathfrak{C}_\Sigma. \tag{10.12}$$

It follows from Lemma 5.10 that $\mathfrak{B}_{\widehat{\Sigma}}$ is the unique contraction $\mathcal{H}(\widehat{\mathfrak{W}}_-^{[\perp]})\to\mathcal{X}$ whose restriction to $\mathcal{H}^0(\widehat{\mathfrak{W}}_-^{[\perp]})$ is given by

$$\mathfrak{B}_{\widehat{\Sigma}}\big(\hat{w}_-+\widehat{\mathfrak{W}}_{\mathrm{past}}^{[\perp]}\big)=x(0), \quad \hat{w}_-(\cdot)\in\widehat{\mathfrak{W}}_{\mathrm{past}}, \tag{10.13}$$

where $(x(\cdot),\mathcal{F}_-^{-1}\hat{w}_-(\cdot))$ is the unique stable externally generated past trajectory of Σ whose signal part is $\mathcal{F}_-^{-1}\hat{w}_-(\cdot)$. By Lemma 5.2, $\mathfrak{C}_{\widehat{\Sigma}}$ is the contraction defined by

$$\mathfrak{C}_{\widehat{\Sigma}}x_0=\left\{\hat{w}_++\mathfrak{W}_{\mathrm{fut}}\;\middle|\;\begin{array}{l}w_+:=\mathcal{F}_+^{-1}\hat{w}_+ \text{ is the signal part of some stable}\\ \text{future trajectory } \big(x(\cdot),w_+(\cdot)\big) \text{ of } \Sigma \text{ with } x(0)=x_0\end{array}\right\}. \tag{10.14}$$

Let $\mathcal{W}=-\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ be a fixed fundamental decomposition of $\mathcal{W}$, and let $\widehat{T}_-:\mathcal{H}(\widehat{\mathfrak{W}}_-^{[\perp]})\to\mathcal{H}(\widehat{\mathfrak{D}}_-^*)$ and $\widehat{T}_+:\mathcal{H}(\widehat{\mathfrak{W}}_+)\to\mathcal{H}(\widehat{\mathfrak{D}}_+)$ be the two unitary operators in (9.19). Under these transformations $\mathfrak{B}_{\widehat{\Sigma}}$ and $\mathfrak{C}_{\widehat{\Sigma}}$ are mapped into the two contractions

$$\begin{aligned} \mathfrak{B}_{\widehat{\Sigma}}^{\widehat{\mathfrak{D}}_-} &= \mathfrak{B}_\Sigma \mathcal{F}_-^{-1} \widehat{T}_-^{-1} : \mathcal{H}(\widehat{\mathfrak{D}}_-^*) \to \mathcal{X}, \\ \mathfrak{C}_{\widehat{\Sigma}}^{\widehat{\mathfrak{D}}_+} &= \widehat{T}_+ \mathcal{F}^+ \mathfrak{C}_\Sigma : \mathcal{X} \to \mathcal{H}(\widehat{\mathfrak{D}}_+). \end{aligned} \tag{10.15}$$

These two maps can be characterized more explicitly in terms of the coefficient matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ of the corresponding scattering i/s/o representation $\Sigma_{\mathrm{i/s/o}} = (\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}))$ of the s/s system Σ. This coefficient matrix is the contraction appearing in the graph representation

$$V = \left\{ \begin{bmatrix} x_1 \\ x_0 \\ y_0 \\ u_0 \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{Y} \\ \mathcal{U} \end{bmatrix} \;\middle|\; \hat{x}_0 \in \mathcal{X},\ u_0 \in \mathcal{U},\ \text{and} \begin{bmatrix} x_1 \\ y_0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right\} \tag{10.16}$$

of the generating subspace V of Σ corresponding to the fundamental decomposition $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$. This means that $(x(\cdot), w(\cdot))$ is a trajectory of Σ on some interval I if and only if $(x(\cdot), u(\cdot), y(\cdot))$ is a trajectory of $\Sigma_{\mathrm{i/s/o}}$ on I, where $w(\cdot) = \left[\begin{smallmatrix} y(\cdot) \\ u(\cdot) \end{smallmatrix}\right]$ and (1.6) holds.

The maps $\mathfrak{B}_{\widehat{\Sigma}}^{\widehat{\mathfrak{D}}_-}$ and $\mathfrak{C}_{\widehat{\Sigma}}^{\widehat{\mathfrak{D}}_-}$ are related to but not identical with the standard input and output maps $\mathfrak{B}_{\Sigma_{\mathrm{i/s/o}}}$ and $\mathfrak{C}_{\Sigma_{\mathrm{i/s/o}}}$ of the i/s/o system Σ. These two maps are defined as follows: If $(x_-(\cdot), u_-(\cdot), y_-(\cdot))$ is a stable externally generated past trajectory of $\Sigma_{\mathrm{i/s/o}}$ then

$$\mathfrak{B}_{\Sigma_{\mathrm{i/s/o}}} u_-(\cdot) = x_-(0),$$

and if $(x_+(\cdot), u_+(\cdot), y_+(\cdot))$ is a (stable) future trajectory of $\Sigma_{\mathrm{i/s/o}}$ with $u_+(\cdot) = 0$, then

$$\mathfrak{C}_{\Sigma_{\mathrm{i/s/o}}} x_+(0) = y_+(\cdot).$$

By using (1.6) one get the following explicit formulas for these two operators:

$$\begin{aligned} \mathfrak{B}_{\Sigma_{\mathrm{i/s/o}}} u_- &= \sum_{k \in \mathbb{Z}^-} A^{-k} B u_-(k), \quad u_- \in \ell^2_-(\mathcal{U}), \\ \mathfrak{C}_{\Sigma_{\mathrm{i/s/o}}} x_0 &= \{ C A^k x_0 \}_{k \in \mathbb{Z}^+}, \quad x_0 \in \mathcal{X}; \end{aligned} \tag{10.17}$$

see, e.g., [12, p. 697]. It follows from (2.8) that $\mathfrak{B}_{\Sigma_{\mathrm{i/s/o}}}$ is a contraction $\ell^2_-(\mathcal{U}) \to \mathcal{X}$, and it follows from (1.8) with $m = 0$ that $\mathfrak{C}_{\Sigma_{\mathrm{i/s/o}}}$ is a contraction $\mathcal{X} \to \ell^2_+(\mathcal{Y})$.

We denote the frequency domain version of $\mathfrak{B}_{\Sigma_{\mathrm{i/s/o}}}$ and $\mathfrak{C}_{\Sigma_{\mathrm{i/s/o}}}$ by

$$\mathfrak{B}_{\widehat{\Sigma}_{\mathrm{i/s/o}}} := \mathfrak{B}_{\Sigma_{\mathrm{i/s/o}}} \mathcal{F}_-^{-1}, \qquad \mathfrak{C}_{\widehat{\Sigma}_{\mathrm{i/s/o}}} := \mathcal{F}_+ \mathfrak{C}_{\Sigma_{\mathrm{i/s/o}}}. \tag{10.18}$$

The operators $\mathfrak{C}_{\widehat{\Sigma}}^{\widehat{\mathfrak{D}}_+}$ and $\mathfrak{C}_{\widehat{\Sigma}_{\mathrm{i/s/o}}}$ are related in the following way.

Lemma 10.1. *The operator $\mathfrak{C}_{\widehat{\Sigma}_{\mathrm{i/s/o}}}$ is the composition of $\mathfrak{C}_{\widehat{\Sigma}}^{\widehat{\mathfrak{D}}_-}$ and the inclusion map $\mathcal{H}(\widehat{\mathfrak{D}}_+) \hookrightarrow H^2_+(\mathcal{Y})$.*

Proof. This follows from (9.19). □

Lemma 10.2. *The operator* $\mathfrak{B}_{\widehat{\Sigma}}^{\widehat{\mathfrak{D}}_-}$ *is the unique linear contraction* $\mathcal{H}(\widehat{\mathfrak{D}}_-^*) \to \mathcal{X}$*, which is defined by the relation*

$$\mathfrak{B}_{\widehat{\Sigma}}^{\widehat{\mathfrak{D}}_-} = \mathfrak{B}_{\widehat{\Sigma}_{\mathrm{i/s/o}}}\big(1 - \widehat{\mathfrak{D}}_-^* \widehat{\mathfrak{D}}_-\big)^{[-1]}, \tag{10.19}$$

on the dense subspace $\mathcal{H}^0(\widehat{\mathfrak{D}}_-^*) = \mathcal{R}(1 - \widehat{\mathfrak{D}}_-^* \widehat{\mathfrak{D}}_-)$ *of* $\mathcal{H}(\widehat{\mathfrak{D}}_-^*)$ *and then extended to* $\mathcal{H}(\widehat{\mathfrak{D}}_-^*)$ *by continuity.*

Proof. The proof of this is a simplified version of the proof of Lemma 9.1, and it is left to the reader. □

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